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THEORY OF ANGULAR MOMENTUM

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Contents

FOREWORD

TRANSLATOR'S NOTE

Chapter I ADDITION OF TWO ANGULAR MOMENTA 1

1 Angular momentum operators and spatial rotations 1
2 Angular momentum eigenfunctions and representations of the rotation group 2
3 Addition of angular momenta, reduction of the direct product of representations of the rotation group 5
4 Expressions for the Clebsch-Gordan coefficients and their properties 8
5. Wigner coefficients and their properties 13

Chapter II ADDITION OF AN ARBITRARY NUMBER OF ANGULAR MOMENTA . . 16

6 General considerations on the addition of an arbitrary number of angular momenta 16
7 Group-theoretic considerations on the generalized Clebsch-Gordan coefficients 19
8 The transformation matrix . . . 21
9. Simplification of the transformation matrix 24
10. Generalized Wigner coefficients and their properties 27

Chapter III GRAPHICAL METHODS FOR OPERATIONS WITH SUMS OF PRODUCTS OF WIGNER COEFFICIENTS 31

11 Sums of products of Wigner coefficients (jm-coefficients) 31
12 Graphical representation of jm-coefficients . 34
13 Expansion of jm-coefficients in generalized Wigner coefficients 39
14 Transformation of jm-coefficients 42
15. Summation of jm-coefficients 46

Chapter IV j-COEFFICIENTS AND THEIR PROPERTIES . 49

16 The 6 j-coefficient and its properties . . . 49
17 $3nj$-coefficients of the first and second kinds 55
18 The 9 j-coefficient and its properties 59
19 12 j-coefficients and their properties . . 62
20 Methods of studying j-coefficients, 15 j-coefficients 65

Chapter V UTILIZATION OF TRANSFORMATION MATRICES FOR OBTAINING SUM RULES AND TRANSFORMATION FORMULAS FOR *jm*-COEFFICIENTS 71

21 General considerations on the relation between transformation matrices and *j*-coefficients 71
22 Methods for obtaining the relation between transformation matrices and *j*-coefficients 74
23. Explicit expressions for the simplest transformation matrices 76
24 Utilization of matrix identities for obtaining sum rules on *j*-coefficients 80
25 Use of matrix identities for the transformation of *jm*-coefficients 83

Chapter VI EXAMPLES OF APPLICATION OF THE GRAPHICAL METHOD 87

26. Graphical summation of products of Wigner coefficients 87
27 A more complex product of Wigner coefficients 91
28 Summation of a product of *j*-coefficients 94
29. Summation of a product of Wigner coefficients and *j*-coefficients 98
30. Choice of a method of calculation 102

Chapter VII. IRREDUCIBLE TENSOR OPERATORS AND EXPRESSIONS FOR THEIR MATRIX ELEMENTS 105

31. Irreducible tensor operators and their properties 105
32. Tensor products 107
33 Expressions for matrix elements of products of tensor operators 109
34 Calculation of matrix elements of complex products of tensor operators 111
35 Double tensors, their products and matrix elements 114

Appendix 1. NOTATIONS FOR THE WIGNER, 6*j*-, 9*j*- AND ALLIED COEFFICIENTS 117

Appendix 2 ALGEBRAIC FORMULAS FOR THE CLEBSCH-GORDAN COEFFICIENTS 119

Appendix 3 DIAGRAMS OF 18*j*-COEFFICIENTS 122

Appendix 4 PROPERTIES OF 18*j*-COEFFICIENTS 126

Appendix 5 EXPRESSIONS FOR THE TRANSFORMATION MATRICES OF EIGENFUNCTIONS OF FIVE COUPLED ANGULAR MOMENTA 135

Appendix 6 SUM RULES ON *j*-COEFFICIENTS 141

Appendix 7 THE SIMPLEST SUMMATION AND TRANSFORMATION FORMULAS FOR *jm*-COEFFICIENTS 152

BIBLIOGRAPHY 154
SUPPLEMENTARY BIBLIOGRAPHY 157
INDEX 158

FOREWORD

The principal results obtained up to 1935 in the quantum-mechanical theory of angular momentum are contained in chapter III of Condon and Shortley's "Theory of Atomic Spectra" /1949/ Since then, owing to the ideas of Wigner /1931, 1937/ and Racah /1942/, the theory has been enriched by the algebra of noncommuting tensor operators and the theory of j-coefficients. This has considerably increased its computational possibilities and has broadened the scope of its applications. Among the branches of theoretical physics where the methods of the theory of angular momentum are widely applied today we might mention the theory of atomic and nuclear spectra, the scattering of polarized particles in nuclear reactions, the theory of genealogical coefficients, etc. (a bibliography of the applications may be found in Edmonds' book /1957/).

The only book known to us giving an exposition of the algebra of noncommuting tensor operators and j-coefficients is Edmonds' "Angular Momentum in Quantum Mechanics" /1957/, which may serve as an excellent textbook for a first acquaintance with the subject. However, the exposition of the theory of j-coefficients and transformation matrices given in this book is not complete. This may constitute an impediment when the apparatus is employed in more complicated cases. The present book fills this gap.

The writing of this book began before Edmonds' book appeared in print. The authors have utilized nearly all results known to them in the given field. Among these a certain place is occupied by the results obtained by a group of workers under the direction of one of the present authors (A. Yutsis), the remaining two authors (I. Levinson and V. Vanagas) being the principal participants. The book corresponds to the content of the first part of a course, "Methods of Quantum-Mechanical Atomic Calculations", given by the senior author to students of theoretical physics at the Vilnius State University im. V. Kapsukas over the last two years.

We found it worthwhile to use the elegant and powerful methods of group theory in our exposition. To avoid encumbering the book with elements of group theory we have assumed that the reader is already acquainted with linear representations of the three-dimensional rotation group. The reader who is unfamiliar with this may refer to the books by G. Ya. Lyubarskii*/1957/ and I. M. Gel'fand et al. /1958/.

We begin with the well-known theory of vector addition of two angular momenta (chapter I), turning next to the addition of an arbitrary number of angular momenta (chapter II). The following chapters (III-VI) are devoted to quantities of the theory

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* English translation by Stevan Dedijer, Pergamon Press, 1960, alternatively, the reader may consult Wigner /1931/ or Racah /1951/ -Translator.

of angular momentum where an important place is occupied by the graphical method which is convenient for various calculations. The last chapter (VII) deals with the method of noncommuting tensor operators. Material of a supplementary character is given in the appendices.

We have cited a number of unpublished works some of which were not available to us. References to these were based on other published works. We apologise in advance for any resulting inaccuracy.

The authors will be grateful for criticism which should be addressed to: Vilnius, ul. Kostsyushkos 30, Institut fiziki i matematiki AN Litovskoi SSR

Vilnius
August, 1959

A. P. Yutsis
I. B. Levinson
V. V. Vanagas

TRANSLATOR'S NOTE

The present book is the translation of a Russian work published in Vilnius and carrying an alternative title page in the Lithuanian language, which is written in Latin script. The translators were therefore faced with an interesting and nontrivial problem, that of deciding whether to spell the names of the authors as they are written in Lithuanian, or as they would be transliterated from the Russian The latter alternative was adopted as being the form which would be most easily recognizable in the literature in Russian, to the best of our knowledge, a large number of works of the authors of this book have been published in Russian, and the Western reader is most likely to encounter references to these in Russian periodicals The Lithuanian version of the authors' names, as given on the Lithuanian title page, is:

A. Jucys
J. Levinsonas
V. Vanagas

This system has led to a certain double-valuedness within the translation itself The reader will find both "Yutsis" and "Jucys" in the text and in the bibliography The first is used to quote references cited in Russian, and the second for references cited in Lithuanian.

The authors' address at the end of the foreword was given only in Cyrillic, we were therefore obliged to render it as a transliteration from the Russian The only exception is the place-name Vilnius, which in transliteration would be Vil'nyus and is known to many as Vilna

If the reader opens the book at random, he is likely to get another surprise, e g "Condon and Shortley 1949" A large number of standard Western works seem to be in circulation in the USSR in Russian translation Several such translations are quoted in the present book. We have in such cases stuck to the book and have referred to the Russian translation. In the bibliography we have also given the original reference, not because any reader of this book will need to be reminded of it, but out of a desire for completeness

In matters of terminology we have occasionally departed from the Russian original We hope that the reader of this translation will not "feel" it, which we also put forward as our justification for taking this liberty We have tried to render the scientific content of the book with the utmost faithfulness However, it is well known that translations, unlike rotations, cannot always be represented in a "unitary" form We therefore do not guarantee that an independent translation of the present book into Russian will recover the original

It should be pointed out that the authors make an essential difference between "Clebsch-Gordan" and "Wigner" coefficients, the latter term being reserved exclusively for the $3j$-symbol. We found it necessary to maintain this distinction also in the translation

The book is divided into sections which are numbered serially throughout the book, and not according to chapters Formulas and diagrams are numbered serially within each section. Thus we might have formula (23 1) or Figure 23 1, referring respectively to the first formula and the first figure in section 23 The numbers of formulas are always in parentheses, that of diagrams, never In the text, therefore, (23.1) would refer to a formula and 23.1 to a figure. The appendices are numbered separately. Corresponding formulas etc. carry the prefix "A".

Several misprints and inaccurate references have been corrected. A small supplementary bibliography and an index have been added. The latter is not comprehensive, but deals essentially with definitions of quantities which are new to Western literature.

Chapter I

ADDITION OF TWO ANGULAR MOMENTA

In this introductory chapter we shall deal with angular momentum and its properties, which is the basis for subsequent development of the mathematical apparatus of this book. Section 1 sketches the connection between angular momentum and the three-dimensional rotation group. Section 2 deals with the corresponding eigenfunctions and representations. The addition of two angular momenta and its relation to the reduction of the direct product of two representations is given in section 3. The Clebsch-Gordan coefficients are examined in section 4, and finally in section 5 the more symmetric Wigner coefficients are introduced.

1. Angular momentum operators and spatial rotations

Let the function ψ describe the state of a physical system in a certain system of cartesian coordinates. If we transform to a new system of coordinates, obtained by rotating the old system through an angle ω about the axis n, the state of the physical system will then be described by a function ψ' which may be obtained from ψ by the operation of an unitary operator $V(n, \omega)$, the latter of course depends on the parameters of the rotation

$$\psi' = V(n, \omega)\psi. \tag{1.1}$$

For $\omega = 0$, V will obviously be the identity operator

Let us write the operator V in the form

$$V(n, \omega) = e^{-i J(n, \omega)}. \tag{1.2}$$

The operator J must tend to zero with ω, and for small ω we therefore have

$$J(n, \omega) = J_n \omega. \tag{1.3}$$

Inserting (1.3) into (1.2) and expanding the exponential in series we obtain, for small ω,

$$V(n, \omega) = 1 - i\omega J_n, \tag{1.4}$$

where 1 denotes the identity operator. For small ω (1.1) therefore becomes

$$\delta\psi \equiv \psi' - \psi = -i\omega J_n \psi. \tag{1.5}$$

Thus, up to the factor $-i$, the operator ωJ_n gives the change in the function ψ upon an infinitesimal rotation about the axis n.

From the unitarity of $\mathbf{V}$ ($\mathbf{V}^+\mathbf{V} = 1$, where $\mathbf{V}^+$ is the hermitian conjugate of $\mathbf{V}$) it follows that the operator $\mathbf{J}_n$ is hermitian. Clearly, from (1.4) we have $(1 + i\omega \mathbf{J}_n^+)(1 - i\omega \mathbf{J}_n) = 1$, whence it follows that $\mathbf{J}_n^+ = \mathbf{J}_n$ (condition of hermiticity). The operator $-i\mathbf{J}_n$ is the operator of infinitesimal rotation about the axis n. The three operators $\mathbf{J}_x$, $\mathbf{J}_y$, $\mathbf{J}_z$ of infinitesimal rotations about the cartesian axes satisfy the well-known commutation relations /Condon and Shortley 1949/

$$[\mathbf{J}_x, \mathbf{J}_y] = i\mathbf{J}_z, \quad [\mathbf{J}_y, \mathbf{J}_z] = i\mathbf{J}_x, \quad [\mathbf{J}_z, \mathbf{J}_x] = i\mathbf{J}_y; \tag{1.6}$$

where [a, b] denotes the commutator ab−ba.

The commutation relations (1.6) are general conditions which are satisfied by the components of an arbitrary angular momentum (measured in units of $h/2\pi$). In particular, they include both the orbital angular momentum $\mathbf{L}$ with components

$$\mathbf{L}_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \text{ (and analogously for } \mathbf{L}_y \text{ and } \mathbf{L}_z) \tag{1.7}$$

and the spin angular momentum with components expressed by the Pauli matrices

$$\mathbf{S}_x = \left\| \begin{matrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{matrix} \right\|, \quad \mathbf{S}_y = \left\| \begin{matrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{matrix} \right\|, \quad \mathbf{S}_z = \left\| \begin{matrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{matrix} \right\|. \tag{1.8}$$

In the first case, the function ψ which describes the state of the system is a function of the space coordinates, in the second case, it is a spinor with two rows and one column.

2. Angular momentum eigenfunctions and representations of the rotation group

We shall use the term "angular momentum eigenfunctions" to denote the eigenfunctions of the operators $\mathbf{J}^2$ and $\mathbf{J}_z$. From the commutation relations (1.6) it may be shown (Condon and Shortley /1929/) that the eigenvalues of the operator $\mathbf{J}^2$ are $j(j+1)$, where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and that the eigenvalues of $\mathbf{J}_z$ are $m = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$. The eigenfunctions of the angular momentum $\mathbf{J}$ may therefore be denoted by $\psi(jm)$. The quantities j and m are either both integral or both half-integral. Further, $|m| \leqslant j$ and for given j, m may assume $2j+1$ values. j is usually called the angular momentum quantum number and m its projection on the z axis or the magnetic quantum number. Operating on the eigenfunctions, $\mathbf{J}^2$ and $\mathbf{J}_z$ give

$$\left.\begin{aligned} \mathbf{J}^2\psi(jm) &= j(j+1)\psi(jm), \\ \mathbf{J}_z\psi(jm) &= m\psi(jm), \\ (\mathbf{J}_x \pm i\mathbf{J}_y)\psi(jm) &= [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}}\psi(jm \pm 1), \end{aligned}\right\} \tag{2.1}$$

These equations determine the standard choice of phases for wave functions with different m.

The last two equations in (2.1) show that, for fixed j, the functions $\psi(jm)$ transform among each other under infinitesimal rotations. Hence these functions transform among each other under finite rotations

$$V(g)\psi(jm) = \sum_{m'} \psi(jm')\left(m' \mid D_j(g) \mid m\right). \tag{2.2}$$

Here g denotes an arbitrary rotation and $V(g)$ the operator of this rotation. The equation (2.2) means that the transformed functions are expressed as linear combinations of the original functions $\psi(jm)$. The coefficients of these linear combinations are elements of the matrix $D_j(g)$ of order $2j+1$ which depends on the parameters of the rotation g. The matrices $D_j(g)$ with different g form an irreducible representation D_j of the rotation group (see, for instance, Wigner /1931/, Gel'fand et al. /1958/, Lyubarskii /1957/), i.e.,

$$D_j(g_1) \cdot D_j(g_2) = D_j(g_1 \cdot g_2), \tag{2.3}$$

or, in terms of the matrix elements,

$$\sum_{m'} \left(m \mid D_j(g_1) \mid m'\right)\left(m' \mid D_j(g_2) \mid m''\right) = \\ = \left(m \mid D_j(g_1 \cdot g_2) \mid m''\right). \tag{2.3a}$$

In accordance with (2.2), the functions $\psi(jm)$ form a basis for the representation D_j.

The *representation* D_j is *unitary*

$$D_j(g^{-1}) = D_j^{-1}(g) = D_j^{+}(g), \tag{2.4}$$

or, which is the same,

$$\left(m \mid D_j(g^{-1}) \mid m'\right) = \left(m \mid D_j^{-1}(g) \mid m'\right) = \left(m' \mid D_j(g) \mid m\right)^{*}. \tag{2.4a}$$

The condition of unitarity may also be written as

$$D_j(g)\, D_j^{+}(g) = 1, \tag{2.5}$$

or, in terms of the matrix elements

$$\sum_{m'} \left(m \mid D_j(g) \mid m'\right)\left(m'' \mid D_j(g) \mid m'\right)^{*} = \delta(m, m''). \tag{2.5a}$$

Further, we have the relation

$$\left(m \mid D_j(g) \mid m'\right)^{*} = (-1)^{m-m'}\left(-m \mid D_j(g) \mid -m'\right) = \\ = (-1)^{(j-m)+(j-m')}\left(-m \mid D_j(g) \mid -m'\right). \tag{2.6}$$

This property of the matrices D_j is related to the following property of the functions ψ

$$\psi^*(jm) = (-1)^{j-m}\psi(j-m). \tag{2.7}$$

Using (2.6), the condition of unitarity may be written in yet another form

$$\sum_{m'}(-1)^{m-m'}\left(m\,|\,D_j(g)\,|\,m'\right)\left(-m''\,|\,D_j(g)\,|\,-m'\right) = \delta(m, m''). \tag{2.5b}$$

The phase factor $(-1)^{m-m'}$ may, if desired, be written in the more symmetrical form $(-1)^{(j-m)+(j-m')}$. The matrices $D_j(g)$ form a system of functions in g, orthogonal on the group of elements g

$$\int\left(m_1\,|\,D_{j_1}(g)\,|\,m_1'\right)\left(m_2\,|\,D_{j_2}(g)\,|\,m_2'\right)\frac{dg}{G} =$$

$$= (2j_1+1)^{-1}\delta(j_1, j_2)(-1)^{j_1-m_1}\,\delta(m_1, -m_2)\times$$

$$\times(-1)^{j_1-m_1'}\delta(m_1', -m_2'). \tag{2.8}$$

Here G is the volume of the rotation group. Noting that $(0\,|\,D_0(g)\,|\,0) = 1$, we obtain from (2.8)

$$\int\left(m\,|\,D_j(g)\,|\,m'\right)\frac{dg}{G} = \delta(j, 0)\,\delta(m, 0)\,\delta(m', 0). \tag{2.9}$$

Here we should add a few remarks concerning the matrices D_j with half-integral j. These matrices do not form representations of the rotation group in the strict sense of the word, as to each rotation g there corresponds two matrices D_j with different signs. This is related to the fact that the wave functions of systems with nonintegral spins change sign upon rotation by 2π about an arbitrary axis. One may avoid this two-valuedness and restrict oneself to usual single-valued representations by formally introducing the so-called double rotation group /Bethe 1929/. In the double rotation group a rotation through 2π about an arbitrary axis is regarded as an element q which is different from the unit element. As a result, the number of elements of the group and the group volume are doubled. When the double rotation group is introduced the matrices D_j with integral and with half-integral j form the usual representations of this group. In all formulas in this section g should be understood as an element of the double rotation group, only in which case will the formulas assume a rigorous meaning for half-integral j. Instead of the double rotation group one can consider a group which is isomorphic to it, namely the group of unimodular matrices*

$$u = \left\| \begin{matrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{matrix} \right\|, \quad \mathrm{Det}\,u = \alpha\alpha^* + \beta\beta^* = 1. \tag{2.10}$$

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* On this point the reader may consult Wigner /1931/, chapter 15 (the section on "The Homomorphism of the Two-Dimensional Unitary Group onto the Rotation Group", p. 157 of the English translation), or "Topological Groups", by Leon Pontrjagin, Princeton University Press, 1939, ch. IX, for a more general discussion.

The explicit form of the matrices D_j, which are expressed in terms of the Euler angles or the Cayley-Klein parameters α and β was obtained by Wigner /1931/. The first of these expressions can only be used for the usual group of rotations and integral j. The second is also suitable for the double rotation group and for half-integral j. The asymptotic expressions for the matrices D_j for large j (semi-classical approximation) can be found in Brussard and Tolhoek /1957/.

3. Addition of angular momenta, reduction of the direct product of representations of the rotation group

The wave functions of a quantum-mechanical system in a spherically symmetric field are the eigenfunctions of a certain angular momentum $\mathbf{J}$ and are characterized by the quantum numbers j and m. We shall denote these functions by $\psi(\alpha jm)$, where α is the set of additional quantum numbers required to complete the set. We note that if a different set of additional quantum numbers β is chosen, the transformation matrix will have the following property /Condon and Shortley 1949/

$$(\alpha jm \mid \beta j'm') = \delta(j,j')\,\delta(m,m')\,(\alpha j \mid \beta j). \tag{3.1}$$

In the absence of an external field, the energy of the system does not depend on m (i.e., on the orientation of the "vector" j in space), and therefore all wave functions with the same α and j belong to the same energy level, which thus has a $(2j+1)$-fold degeneracy.

Let there be two weakly interacting systems. In the zeroeth approximation, i.e., when the interaction between the systems is neglected, the wave function of the coupled system is the product of the wave functions of its component parts

$$\varphi(\alpha_1 j_1 \alpha_2 j_2; m_1 m_2) = \psi(\alpha_1 j_1 m_1)\,\psi(\alpha_2 j_2 m_2). \tag{3.2}$$

and the energy of the coupled system, being the sum of energies of the component parts, depends only on $\alpha_1 j_1 \alpha_2 j_2$. Consequently, all the $f = (2j_1+1)(2j_2+1)$ wave functions (3.2) with different m_1 and m_2 belong to the same energy level, which is f-fold degenerate. To calculate the interaction in the first approximation we must, as is known, set up the proper wave functions ψ in the zeroeth approximation which are linear combinations of the wave functions φ of the zeroeth approximation belonging to the same energy level. The proper wave functions in the zeroeth approximation should be the eigenfunctions of the operator $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ with certain quantum numbers j and m. This requirement can be satisfied by choosing the coefficients of the linear combination to be independent of α_1 and α_2. Thus we have

$$\psi(\alpha_1 j_1 \alpha_2 j_2 jm) =$$

$$= \sum_{m_1 m_2} \psi(\alpha_1 j_1 m_1)\,\psi(\alpha_2 j_2 m_2)\,(j_1 j_2 m_1 m_2 \mid j_1 j_2 jm). \tag{3.3}$$

The construction of eigenfunctions of the operator $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ from the eigenfunctions of the operators $\mathbf{J}_1$ and $\mathbf{J}_2$ described by the equation (3.3) will be

termed the addition or coupling of the angular momenta j_1 and j_2 to the angular momentum j, and the wave function (3.3) the wave function of two coupled angular momenta. Further, we assume that the eigenfunctions of each angular momentum do not undergo transformations of the type (3.1) upon coupling. In such cases we shall simply omit the quantum numbers α_1 and α_2. If the coupling is accompanied by a transformation of the eigenfunctions of the individual angular momenta, then, since from (3.1) the matrices of these transformations are independent of the magnetic quantum numbers, we have

$$(\alpha_1 j_1 \alpha_2 j_2 m_1 m_2 \mid \beta_1 j_1 \beta_2 j_2 jm) =$$

$$= (\alpha_1 j_1 \mid \beta_1 j_1)(\alpha_2 j_2 \mid \beta_2 j_2)(j_1 j_2 m_1 m_2 \mid j_1 j_2 jm). \tag{3.4}$$

This equation shows that the transformations of the wave functions of the individual angular momenta are independent of the coupling. The inverse transformation from the functions (3.3) to the functions (3.2) is given by

$$\psi(\alpha_1 j_1 m_1)\psi(\alpha_2 j_2 m_2) =$$

$$= \sum_{jm} \psi(\alpha_1 j_1 \alpha_2 j_2 jm)(j_1 j_2 jm \mid j_1 j_2 m_1 m_2). \tag{3.5}$$

The coupling of angular momenta can be regarded as the transformation of an orthonormal system of functions $\varphi(m_1 m_2)$ belonging to a given energy level into another orthonormal system $\psi(jm)$ belonging to the same level. The coefficients of the expansion (3.3) should then be understood as the elements $(m_1 m_2 \mid C_{j_1 j_2} \mid jm)$ of an unitary matrix $C_{j_1 j_2}$ which effects this transformation. It is obvious that the coefficients of the expansion (3.5) will then be the elements $(jm \mid C^{-1}_{j_1 j_2} \mid m_1 m_2)$ of the inverse matrix $C^{-1}_{j_1 j_2}$. The matrix C may be chosen to be real. The coefficients of the expansions (3.3) and (3.5) will then be identical. This means that

$$(j_1 j_2 m_1 m_2 \mid j_1 j_2 jm) = (j_1 j_2 jm \mid j_1 j_2 m_1 m_2). \tag{3.6}$$

These coefficients are called the vector addition or Clebsch-Gordan coefficients.

The conditions of unitarity of the matrix C reduce to the following orthogonality conditions of the Clebsch-Gordan coefficients

$$\sum_{m_1 m_2} (j_1 j_2 jm \mid j_1 j_2 m_1 m_2)(j_1 j_2 m_1 m_2 \mid j_1 j_2 j'm') = \delta(j, j')\delta(m, m'), \tag{3.7a}$$

$$\sum_{jm} (j_1 j_2 m_1 m_2 \mid j_1 j_2 jm)(j_1 j_2 jm \mid j_1 j_2 m_1' m_2') =$$

$$= \delta(m_1, m_1')\delta(m_2, m_2'). \tag{3.7b}$$

The Clebsch-Gordan coefficients are non-zero only when the following conditions are fulfilled

$$j_1 + j_2 \geq j \geq |j_1 - j_2|, \tag{3.8a}$$

$$j_1 + j_2 + j = \text{integer}, \tag{3.8b}$$

$$m_1 + m_2 = m. \tag{3.8c}$$

These are the so-called integral perimeter triangle conditions or simply the triangle conditions.

The Clebsch-Gordan coefficients are closely related to the representations of the rotation group. The $(2j_1+1)(2j_2+1)$ functions (3.2), in which j_1 and j_2 are independent, form the basis of the direct (Kronecker) product representation $D_{j_1} \times D_{j_2}$ of the rotation group. In fact

$$V(g)\varphi(\alpha_1 j_1 \alpha_2 j_2; m_1 m_2) =$$

$$= \sum_{m_1' m_2'} \varphi(\alpha_1 j_1 \alpha_2 j_2; m_1' m_2') \left(m_1' m_2' \mid D_{j_1}(g) \times D_{j_2}(g) \mid m_1 m_2\right), \tag{3.9}$$

where the matrix elements of the direct product are

$$\left(m_1' m_2' \mid D_{j_1}(g) \times D_{j_2}(g) \mid m_1 m_2\right) =$$

$$= \left(m_1' \mid D_{j_1}(g) \mid m_1\right)\left(m_2' \mid D_{j_2}(g) \mid m_2\right). \tag{3.10}$$

The direct product $D_{j_1} \times D_{j_2}$ is reducible, this means that there exists a matrix $C_{j_1 j_2}$ such that

$$C_{j_1 j_2}^{-1} \cdot (D_{j_1} \times D_{j_2}) \cdot C_{j_1 j_2} = D_{j_1+j_2} \dot{+} \cdots \dot{+} D_{|j_1-j_2|}. \tag{3.11}$$

Here the right-hand side contains a 'quasi-diagonal' matrix with the submatrices D_j, $j = j_1+j_2, \ldots, |j_1-j_2|$, further, each submatrix D_j appears only once. The wave functions of the coupled angular momenta form the basis of the reduced representation, and the matrix $C_{j_1 j_2}$ is identical with the above-mentioned matrix. The Clebsch-Gordan coefficients are thus the elements of the matrix which reduces the direct product of two irreducible representations of the rotation group /Wigner 1931/. In terms of the matrix elements equation (3.11) therefore assumes the form

$$\sum_{\substack{m_1 m_2 \\ m_1' m_2'}} (j_1 j_2 j m \mid j_1 j_2 m_1 m_2)\left(m_1 \mid D_{j_1}(g) \mid m_1'\right)\left(m_2 \mid D_{j_2}(g) \mid m_2'\right) \times$$

$$\times (j_1 j_2 m_1' m_2' \mid j_1 j_2 j' m') = \delta(j, j')\left(m \mid D_j(g) \mid m'\right). \tag{3.11a}$$

From the unitarity of the Clebsch-Gordan coefficients (3.7), we obtain the inverse relation

$$\left(m_1 \mid D_{j_1}(g) \mid m_1'\right)\left(m_2 \mid D_{j_2}(g) \mid m_2'\right) =$$

$$= \sum_{jmm'} (j_1 j_2 m_1 m_2 \mid j_1 j_2 j m)\left(m \mid D_j(g) \mid m'\right)(j_1 j_2 j m' \mid j_1 j_2 m_1' m_2'). \tag{3.11b}$$

Multiplying this equation by $(m_3 \mid D_{j_3}(g) \mid m_3')$ and integrating over the group according to (2.8) we obtain the relation

$$\int \left(m_1 \mid D_{j_1}(g) \mid m_1'\right)\left(m_2 \mid D_{j_2}(g) \mid m_2'\right)\left(m_3 \mid D_{j_3}(g) \mid m_3'\right)\frac{dg}{G} =$$

$$= (-1)^{j_3 - m_3 + j_3 - m_3'}(2j_3+1)^{-1}(j_1 j_2 m_1 m_2 \mid j_1 j_2 j_3 - m_3) \times$$

$$\times (j_1 j_2 m_1' m_2' \mid j_1 j_2 j_3 - m_3'). \tag{3.12}$$

In this equation we substitute gg' for g and replace each matrix element by the expression

$$\left(m \mid D_j(gg') \mid m'\right) = \sum_{m''} \left(m \mid D_j(g) \mid m''\right)\left(m'' \mid D_j(g') \mid m'\right). \tag{3.13}$$

Integrating the equation thus obtained over the group we find, according to (3.12),

$$\sum_{m_1'' m_2'' m_3''} (-1)^{j_3 - m_3''} \left(m_1'' \mid D_{j_1}(g) \mid m_1'\right)\left(m_2'' \mid D_{j_2}(g) \mid m_2'\right) \times$$

$$\times \left(m_3'' \mid D_{j_3}(g) \mid m_3'\right) \left(j_1 j_2 m_1'' m_2'' \mid j_1 j_2 j_3 - m_3''\right) \times$$

$$\times \left(j_1 j_2 m_1 m_2 \mid j_1 j_2 j_3 - m_3\right) = (-1)^{j_3 - m_3}\left(j_1 j_2 m_1 m_2 \mid j_1 j_2 j_3 - m_3\right) \times$$

$$\times \left(j_1 j_2 m_1' m_2' \mid j_1 j_2 j_3 - m_3'\right). \tag{3.14}$$

If m_1, m_2 and m_3 are so chosen that the common factor in this equation is not zero then, dividing by the Clebsch-Gordan coefficient and dropping one prime everywhere, we obtain the following equation

$$(-1)^{j_3 - m_3}\left(j_1 j_2 m_1 m_2 \mid j_1 j_2 j_3 - m_3\right) =$$

$$= \sum_{m_1' m_2' m_3'} (-1)^{j_3 - m_3'}\left(j_1 j_2 m_1' m_2' \mid j_1 j_2 j_3 - m_3'\right) \times$$

$$\times \left(m_1' \mid D_{j_1}(g) \mid m_1\right)\left(m_2' \mid D_{j_2}(g) \mid m_2\right)\left(m_3' \mid D_{j_3}(g) \mid m_3\right), \tag{3.15}$$

which will be useful later on.

4. Expressions for the Clebsch-Gordan coefficients and their properties

The Clebsch-Gordan coefficients are frequently encountered in quantum-mechanical calculations. It is therefore useful to have explicit expressions for them in terms of the parameters j_1, j_2, j, m_1, m_2, m. Such expressions were obtained in different ways by Wigner /1931/, Van der Waerden /1938/, Racah /1942/ and Majumdar /1958/.

Wigner used an explicit form of the matrices D_j and obtained the Clebsch-Gordan coefficients from (3.12). Van der Waerden's method consisted of constructing, with the help of spinors, a space in which D_{j_1}, D_{j_2} and D_j operate (spinors are quantities which transform according to the representation $D_{\frac{1}{2}}$). Racah made use of the recursion relations for Clebsch-Gordan coefficients. Finally, Majumdar obtained an equation which related the Clebsch-Gordan coefficients to the hypergeometric series, from which the required expression could be found.

The expressions for the Clebsch-Gordan coefficients obtained by the four methods described above are equivalent, although they differ in their outward form.

Each possesses its own advantages in particular cases. We shall therefore cite the corresponding formulas obtained by all four methods.

Wigner's formula

$$(j_1 j_2 m_1 m_2 | j_1 j_2 jm) = \delta(m_1 + m_2, m) \cdot \Delta(j_1 j_2 j) \times$$
$$\times \left[\frac{(j+m)!(j-m)!(2j+1)}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!} \right]^{\frac{1}{2}} \times$$
$$\times \sum_z (-1)^{j_2+m_2+z} \frac{(j+j_2+m_1-z)!(j_1-m_1+z)!}{z!(j+m-z)!(j-j_1+j_2-z)!(j_1-j_2-m+z)!}. \qquad (4.1)$$

Van der Waerden's formula

$$(j_1 j_2 m_1 m_2 | j_1 j_2 jm) = \delta(m_1 + m_2, m) \cdot \Delta(j_1 j_2 j) \times$$
$$\times \left[(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)! \times \right.$$
$$\left. \times (j+m)!(j-m)!(2j+1) \right]^{\frac{1}{2}} \times$$
$$\times \sum_z (-1)^z \frac{1}{z!(j_1+j_2-j-z)!(j_1-m_1-z)!(j_2+m_2-z)! \times} \cdot \qquad (4.2)$$
$$\times (j-j_2+m_1+z)!(j-j_1-m_2+z)!$$

Racah's formula

$$(j_1 j_2 m_1 m_2 | j_1 j_2 jm) = \delta(m_1 + m_2, m) \cdot \Delta(j_1 j_2 j) \times$$
$$\times \left[\frac{(j_1-m_1)!(j_2-m_2)!(j+m)!(j-m)!(2j+1)}{(j_1+m_1)!(j_2+m_2)!} \right]^{\frac{1}{2}} \times$$
$$\times \frac{(-1)^{j_1-m_1}}{(j+j_1-j_2)!(j+j_2-j_1)!} \times$$
$$\times \sum_z (-1)^z \frac{(j_1+m_1+z)!(j+j_2-m_1-z)!}{z!(j-m-z)!(j_1-m_1-z)!(j_2-j+m_1+z)!}. \qquad (4.3)$$

Majumdar's formula

$$(j_1 j_2 m_1 m_2 | j_1 j_2 jm) = \delta(m_1 + m_2, m) \Delta(j_1 j_2 j) \times$$
$$\times \left[\frac{(j_1+m_1)!(j_1-m_1)!(j_2-m_2)!(j+m)!(2j+1)}{(j_2+m_2)!(j-m)!} \right]^{\frac{1}{2}} \times$$
$$\times \frac{(-1)^{j_2+m_2}}{(j_1-j_2+j)!(j_1+j_2-j)!} \times$$
$$\times \sum_z (-1)^z \frac{(2j-z)!(j_1+j_2-j+z)!}{z!(j+m-z)!(-j_1+j_2+j-z)!(j_1-j-m_2+z)!}. \qquad (4.4)$$

In these formulas

$$\Delta(j_1 j_2 j_3) = \left[\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!} \right]^{\frac{1}{2}} \qquad (4.5)$$

is the so-called triangle coefficient. The sums in the formulas (4.1) to (4.4) are essentially hypergeometric series, owing to which the Clebsch-Gordan coefficients

can be expressed in terms of the hypergeometric function /Rose 1955/

$$(j_1 j_2 m_1 m_2 | j_1 j_2 jm) = \delta(m_1 + m_2, m) \cdot (-1)^{j_2 + m_2} \times$$

$$\times \left[\frac{(j + j_1 - j_2)!(j_1 + j_2 - j)!(j - m)!(j_1 - m_1)!(2j+1)}{(j - j_1 + j_2)!(j_1 + j_2 + j + 1)!(j + m)!(j_1 + m_1)!(j_2 - m_2)!(j_2 + m_2)!} \right]^{\frac{1}{2}} \times$$

$$\times \frac{(j + j_2 + m_1)!}{(j_1 - j_2 - m)!} \cdot {}_3F_2(-j + j_1 - j_2, j_1 - m_1 + 1,$$

$$-j - m; j_1 - j_2 - m + 1, \; -j - j_2 - m_1; 1). \tag{4.6}$$

Using the explicit expressions of the Clebsch-Gordan coefficients, it is possible to obtain the following symmetries /Racah 1942/

$$(j_1 j_2 m_1 m_2 | j_1 j_2 jm) = \tag{4.7}$$

$$= (-1)^{j - j_1 - m_2} \left[\frac{2j+1}{2j_1 + 1} \right]^{\frac{1}{2}} (j_2 j m_2 - m | j_2 j j_1 - m_1) = \qquad \text{(a)}$$

$$= (-1)^{j - j_2 + m} \left[\frac{2j+1}{2j_2 + 1} \right]^{\frac{1}{2}} (j j_1 - m m_1 | j j_1 j_2 - m_2) = \qquad \text{(b)}$$

$$= (-1)^{j_1 + j_2 - j} (j_2 j_1 m_2 m_1 | j_2 j_1 jm) = \qquad \text{(c)}$$

$$= (-1)^{j_1 - m_1} \left[\frac{2j+1}{2j_2 + 1} \right]^{\frac{1}{2}} (j_1 j m_1 - m | j_1 j j_2 - m_2) = \qquad \text{(d)}$$

$$= (-1)^{j_1 + m_2} \left[\frac{2j+1}{2j_1 + 1} \right]^{\frac{1}{2}} (j j_2 - m m_2 | j j_2 j_1 - m_1) = \qquad \text{(e)}$$

$$= (-1)^{j_1 + j_2 - j} (j_1 j_2 - m_1 - m_2 | j_1 j_2 j - m). \tag{4.8}$$

Combining (4.7) and (4.8) we obtain all the symmetry properties of the Clebsch-Gordan coefficients. It is convenient to transform the sums (4.1)-(4.4) with the aid of the symmetry relations so as to be left with the least number of terms in them.

It should be noted that use of the symmetry properties and replacement of the summation parameter does not transform one expression for the Clebsch-Gordan coefficients into a different one. Transforming from one form into another requires rather complicated algebraic manipulations, such as were carried out, for instance, by Racah /1942/ to transform (4.3) into (4.2).

In certain cases the sums in (4.1)-(4.4) can be reduced to a single term. This happens when the triangle (j_1, j_2, j_3) is "stretched", i.e., for $j_1 + j_2 = j$, or when one of the parameters j_1, j_2, j_3 is zero. It is not difficult to verify that

$$(j_1 0 m_1 0 | j_1 0 jm) = \delta(j_1 j) \delta(m_1, m), \tag{4.9a}$$

$$(j_1 j_2 m_1 m_2 | j_1 j_2 00) = \delta(j_1, j_2) \delta(m_1, -m_2) \times$$

$$\times (-1)^{j_1 - m_1} (2j_1 + 1)^{-\frac{1}{2}}, \tag{4.9b}$$

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j_1 + j_2 m_1 + m_2) =$$

$$= \left[\frac{(2j_1)!(2j_2)!(j_1+j_2+m_1+m_2)!(j_1+j_2-m_1-m_2)!}{(2j_1+2j_2)!(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!} \right]^{\frac{1}{2}}. \qquad (4.10)$$

A single-term formula for coefficients of the form $(j_1 j_1 m_1 m_1 | j_1 j_1 j\ 2m_1)$ has been given by Stone /1956/. With its help single-term formulas may be obtained for $(j_1 + s\ j_1\ m_1 + \mu\ m_1 | j_1 + s\ j_1\ J\ 2m + \mu)$ for $s = \frac{1}{2}; 1$ and $\mu = \pm\frac{1}{2}; 0, \pm 1$ respectively /Stone 1957/. Racah has shown /1942/ that single-term formulas may also be obtained for the cases in which all magnetic quantum numbers are zero. If the sum of the angular momentum quantum numbers is even, i.e., $l_1 + l_2 + l = 2g$, then

$$(l_1 l_2 00 | l_1 l_2 l 0) =$$

$$= (-1)^{g+l} (2l+1)^{\frac{1}{2}} \Delta(l_1 l_2 l) \frac{g!}{(g-l_1)!(g-l_2)!(g-l)!}. \qquad (4.11)$$

If $l_1 + l_2 + l = 2g + 1$, then these Clebsch-Gordan coefficients vanish.

The Clebsch-Gordan coefficients satisfy certain recursion relations in the angular momentum quantum numbers and their projections. The recursion relation in the projections is /Condon and Shortley 1949, Racah 1942/

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m \pm 1) = \left[(j \mp m)(j \pm m + 1) \right]^{\frac{1}{2}} \times$$

$$\times \left\{ \left[(j_1 \mp m_1 + 1)(j_1 \pm m_1) \right]^{\frac{1}{2}} (j_1 j_2\ m_1 \mp 1\ m_2 | j_1 j_2 j m) + \right.$$

$$\left. + \left[(j_2 \mp m_2 + 1)(j_2 \pm m_2) \right]^{\frac{1}{2}} (j_1 j_2 m_1\ m_2 \mp 1 | j_1 j_2 j m) \right\}. \qquad (4.12)$$

The recursion relations in the angular momentum quantum numbers are /Condon and Shortley 1949, Wigner 1937, Louck 1958/

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m) =$$

$$= \left[(2j+1) / (j_1 + j_2 + j + 1)(-j_1 + j_2 + j)(j_1 - j_2 + j)(j_1 + j_2 - j - 1) \right]^{\frac{1}{2}} \times$$

$$\times \left\{ \left[\frac{(j_2 - m_2 + 1)(j_2 + m_2)(j + m - 1)(j + m)}{(2j-1)} \right]^{\frac{1}{2}} (j_1 j_2\ m_1 - 1\ m_2 + 1 | j_1 j_2\ j - 1\ m - 1) + \right.$$

$$+ \left[\frac{4(j-m)(j+m) m_2^2}{(2j-1)} \right]^{\frac{1}{2}} (j_1 j_2\ m_1 m_2 | j_1 j_2\ j - 1\ m) +$$

$$+ \left[\frac{(j_2 + m_2 + 1)(j_2 - m_2)(j - m - 1)(j - m)}{(2j-1)} \right]^{\frac{1}{2}} \times$$

$$\left. \times (j_1 j_2 m_1\ m_2 + 1 | j_1 j_2\ j - 1\ m + 1) \right\}, \qquad (4.13)$$

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m) =$$

$$= \left[(2j+1)/(j_1+j_2+j+1)(-j_1+j_2+1)(2j) \right]^{\frac{1}{2}} \times$$

$$\times \left\{ \left[(j_2 - m_2)(j-m) \right]^{\frac{1}{2}} \left(j_1 j_2 - \tfrac{1}{2}\, m_1\, m_2 + \tfrac{1}{2} \right| \right.$$

$$\left. \left| j_1 j_2 - \tfrac{1}{2}\, j - \tfrac{1}{2}\, m + \tfrac{1}{2} \right) + \right.$$

$$+ \left[(j_2 + m_2)(j+m) \right]^{\frac{1}{2}} \left(j_1 j_2 - \tfrac{1}{2}\, m_1\, m_2 - \tfrac{1}{2} \right|$$

$$\left. \left| j_1 j_2 - \tfrac{1}{2}\, j - \tfrac{1}{2}\, m - \tfrac{1}{2} \right) \right\}. \qquad (4.14)$$

The symmetry properties and recursion formulas for the Clebsch-Gordan coefficients can be reduced to a more convenient form by considering the symmetrical part of the coefficients, the so-called Wigner coefficients. These coefficients will be examined in section 5, where we also give a few additional properties of the Wigner coefficients which can, if desired, be rewritten for the Clebsch-Gordan coefficients.

Ten-figure decimal tables of numerical values of the Clebsch-Gordan coefficeints for the parameters

$$j_1 = 0(1)4;\ j_2, j = 0\left(\tfrac{1}{2}\right)\tfrac{9}{2}$$

have been compiled by Simon /1954/. Here $a(c)b$ means that the parameter varies from a to b in steps of c. Less complete tables can also be found in Kumar /1957/ and Alder /1952/.

In some cases tables of algebraic formulas for certain values of one parameter are also used. For example, if the value of j_2 is given, j can assume the values from $|j_1 - j_2|$ to $j_1 + j_2$ for all possible values of m_2 from $-j_2$ to j_2. Zukauskas and Mauza /1957/ found that in the compilation of such tables it is convenient to use the following formulas obtained from (4.1) with the help of the symmetry properties (4.7)

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m) =$$

$$= A \sum_z (-1)^{j_2 + u + z} \frac{(j_1 + j_2 - m - z)!\,(j + m + z)!}{z!\,(j_2 - m_1 - z)!\,(2j_2 - u - z)!\,(u - j_2 + m_2 + z)!} = \qquad (4.15a)$$

$$= A \sum_z (-1)^{j_1 + m_2 + u + z} \frac{(j + j_2 + m - m_2 - z)!\,(j_1 - m + m_2 + z)!}{z!\,(j_2 - m_2 - z)!\,(u - z)!\,(j_1 + m_2 - u + z)!}. \qquad (4.15b)$$

Here

$$A = \Delta(j_1 j_2 j) \left[\frac{(j_2 + m_2)!\,(j_2 - m_2)!\,(2j+1)}{(j+m)!\,(j-m)!\,(j_1 + m - m_2)!\,(j_1 - m + m_2)!} \right]^{\frac{1}{2}}$$

and

$$u = j - j_1 + j_2.$$

The convenience of these formulas consists of the fact that the number of terms in the sums is small and that after expressing j in terms of j_1, no factorials dependent on j_1 and m remain in the denominator under the summation sign.

The algebraic expressions for the Clebsch-Gordan coefficients for the cases $j_2 = \frac{1}{2}$, 1, $\frac{3}{2}$ and 2 are given in Condon and Shortley's book /1949/. For cases where further $j_2 = \frac{5}{2}$ and 3 they are given in Sears and Radtke /1954/. For $j_2 = \frac{5}{2}$ the expressions were obtained by the authors, for $j_2 = 3$ they were taken from Falkoff et al. /1952/. The expressions for $j_2 = \frac{5}{2}$ were also obtained by Saito and Morita /1955/ and Melvin and Swamy /1957/, and for $j_2 = 3$ by Yamada and Morita /1952/. For $j_2 = \frac{7}{2}$ analogous expressions were obtained by Zukauskas and Mauza /1957/. In all these cases the Clebsch-Gordan coefficients are expressed in terms of j_1 and m for given j_2 and m_2, except in Sears and Radtke (loc. cit.) where they have been expressed in terms of j and m for given values of j_1 and m_1.

An examination of the above-mentioned formulas shows that we can write the Clebsch-Gordan coefficients in the following form

$$(j_1 j_2 m - m_2 m_2 | j_1 j_2 j_1 + k m) = A_{j_1 k}(j_1)\, B_{j_1 k, m_2}(j_1 m). \tag{4.16}$$

Here

$$A_{j_1 k}(j_1) = \prod_{\substack{x=-j_2 \\ x \neq k}}^{j_2} (2j_1 + k + 1 + x)^{-\frac{1}{2}}, \tag{4.17}$$

where k varies from $-j_2$ to $+j_2$. The quantity B satisfies the following symmetry condition

$$B_{j_1 k, -m_2}(j_1, m) = (-1)^{j_2 - k}\, B_{j_1 k, m_2}(j_1, -m). \tag{4.18}$$

This property makes it possible to reduce the size of the algebraic tables by nearly one-half, however, it was not used in the compilation of the tables listed above Owing to the availability of the general expression (4.17) for A, it is possible to restrict oneself to giving the expressions for B alone (say, for $m_2 \geqslant 0$) The corresponding tables for existing cases $j_2 \leqslant 4$ are given at the end of the book (appendix 2). Asymptotic expressions for the Clebsch-Gordan coefficients for large values of the parameters may be found in Brussard and Tolhoek /1957/.

5. Wigner coefficients and their properties

The symmetry properties of the Clebsch-Gordan coefficients assume a far more convenient form if we introduce the Wigner coefficients /Wigner 1937/

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{5.1}$$

which are related to the Clebsch-Gordan coefficients by the following formulas

$$(j_1 j_2 j m | j_1 j_2 m_1 m_2) = (-1)^{j_1 - j_2 + j} (j)^{\frac{1}{2}} (-1)^{j-m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}, \tag{5.2a}$$

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m) =$$

$$= (-1)^{j_1 - j_2 - j} (j)^{\frac{1}{2}} (-1)^{j_1 - m_1 + j_2 - m_2} \begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & m \end{pmatrix}. \qquad (5.2b)$$

Here and hereafter (j) will denote $2j+1$.

In accordance with the conditions of nonvanishing of the Clebsch-Gordan coefficients (3.8), the Wigner coefficients are nonzero when the parameters of the upper row j_1, j_2, j_3 form a triad, i.e., can be the sides of a triangle with an integral perimeter, and when the sum of the parameters m_1, m_2, m_3 of the lower row is zero. The parameters $j_1, j_2, j_3, m_1, m_2, m_3$ can be integral or half-integral, further, j_i and m_i are simultaneously integral or half-integral, so that each of the nine numbers

$$\begin{aligned} &j_1 + m_1, \; j_1 - m_1, \quad -j_1 + j_2 + j_3, \\ &j_2 + m_2, \; j_2 - m_2, \quad j_1 - j_2 + j_3, \\ &j_3 + m_3, \; j_3 - m_3, \quad j_1 + j_2 - j_3, \end{aligned} \qquad (5.3)$$

is an integer.

From (4.7) one can easily obtain the following permutation properties of the Wigner coefficients

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \varepsilon \begin{pmatrix} j_i & j_k & j_l \\ m_i & m_k & m_l \end{pmatrix}, \qquad (5.4)$$

where $\varepsilon = 1$ if the permutation $\begin{pmatrix} 1 & 2 & 3 \\ i & k & l \end{pmatrix}$ is even and $\varepsilon = (-1)^{j_1 + j_2 + j_3}$ if it is odd.

From (4.8) it follows that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1 + j_2 + j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \qquad (5.5)$$

The orthogonality properties of the Clebsch-Gordan coefficients (3.7) give the following orthogonality properties of the Wigner coefficients

$$\sum_{m_1 m_2} (-1)^{j_1 - m_1 + j_2 - m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ -m_1 & -m_2 & -m_3' \end{pmatrix} =$$

$$= \delta(j_3, j_3') (j_3)^{-1} \cdot \delta(m_3, m_3') (-1)^{j_3 - m_3} \cdot \{ j_1 j_2 j_3 \} \qquad (5.6a)$$

and

$$\sum_{j_3 m_3} (j_3) (-1)^{j_3 - m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1' & -m_2' & -m_3 \end{pmatrix} =$$

$$= (-1)^{j_1 - m_1} \delta(m_1, m_1') \cdot (-1)^{j_2 - m_2} \delta(m_2, m_2'). \qquad (5.6b)$$

$\{ j_1 j_2 j_3 \}$ is a triangular delta which is equal to unity if j_1, j_2, j_3 form a triad and zero otherwise. The forms of (5.6a) and (5.6b) were chosen for convenience in the subsequent considerations.

From (4.9b) it follows that

$$\begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} \equiv \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix} =$$

$$= \delta(j_1, j_2)(j_1)^{-\frac{1}{2}} \cdot (-1)^{j_1 - m_1} \delta(m_1, -m_2). \tag{5.7}$$

Using (5.2) one can write explicit expressions and recursion formulas for the Wigner coefficients. We note that by using the symmetries one can reduce the sums of the form (4.1)-(4.4) to a form in which the number of terms in the sum over z is only larger by one than the smallest of the nine numbers in (5.3).

In the frequently-encountered particular case $m_1 = m_2 = m_3 = 0$, it follows from (4.11) that

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \equiv (l_1 l_2 l_3) = (-1)^g \Delta(l_1 l_2 l_3) \frac{g!}{(g-l_1)!\,(g-l_2)!\,(g-l_3)!} \tag{5.8a}$$

for even $l_1 + l_2 + l_3 = 2g$ and

$$(l_1 l_2 l_3) = 0 \tag{5.8b}$$

for odd $l_1 + l_2 + l_3$. We note that the quantity $(l_1 l_2 l_3)$ is fully symmetric in all parameters.

We give yet another property of the Wigner coefficients which was obtained by Regge /1958/. For convenience an intermediate notation is introduced

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} -j_1 + j_2 + j_3 & j_1 - j_2 + j_3 & j_1 + j_2 - j_3 \\ j_1 - m_1 & j_2 - m_2 & j_3 - m_3 \\ j_1 + m_1 & j_2 + m_2 & j_3 + m_3 \end{pmatrix}. \tag{5.9}$$

According to (5.3) all the nine numbers in the right-hand side are positive integers. The sum of the numbers in each row and column is $j_1 + j_2 + j_3$. In this notation one can arbitrarily permute the rows or columns, as well as transpose them relative to the principal diagonal. An even permutation of the rows and columns and transposition does not change the Wigner coefficient. Upon an odd permutation of the rows or columns the Wigner coefficient is multiplied by $(-1)^{j_1 + j_2 + j_3}$. The permutation of columns is equivalent to (5.4), and the permutation of the two lower rows to (5.5). The other permutations and transposition give essentially new symmetries.

Replacing the Clebsch-Gordan coefficients in (3.15) by the Wigner coefficients, we obtain the following important relation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \sum_{m_1' m_2' m_3'} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix} \left(m_1' \middle| D_{j_1}(g) \middle| m_1\right) \left(m_2' \middle| D_{j_2}(g) \middle| m_2\right) \times$$

$$\times \left(m_3' \middle| D_{j_3}(g) \middle| m_3\right). \tag{5.10}$$

Instead of the Wigner coefficients, certain authors use other quantities related to them (see Appendix 1). All of these are less symmetrical than the Wigner coefficients, and are therefore much less convenient to use.

Chapter II

ADDITION OF AN ARBITRARY NUMBER OF ANGULAR MOMENTA

The addition of two angular momenta, examined in the preceding chapter, may be immediately generalized to the addition of an arbitrary number of angular momenta. The present chapter is devoted to the study of this problem. In section 6 we examine the general questions concerning the addition of angular momenta and introduce the generalized Clebsch-Gordan coefficients. In the following section we give the connection between the reduction of the direct product of an arbitrary number of representations of the rotation group and the construction of wave functions of the coupled angular momenta.

Upon passing from two to a greater number of angular momenta, the resultant angular momentum and magnetic quantum numbers are no longer sufficient for a complete characterization of the wave function owing to the appearance of intermediate angular momenta. Furthermore, the wave functions of the coupled angular momenta depend upon the scheme of addition. The matrices of transformations between wave functions belonging to different schemes of addition of the angular momenta are discussed in section 8. The next section deals with the problem of simplifying these matrices.

The last section of this chapter introduces a more symmetric quantity than the generalized Clebsch-Gordan coefficient, in the same manner as was done in section 5 for the case of two angular momenta.

6. General considerations on the addition of an arbitrary number of angular momenta

Let us first consider three angular momentum operators $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_3$. We shall denote an eigenfunction of the commuting set of independent operators

$$\mathbf{J}_1^2, \mathbf{J}_2^2, \mathbf{J}_3^2, J_{1z}, J_{2z}, J_{3z} \tag{6.1}$$

by

$$\varphi(j_1 j_2 j_3 m_1 m_2 m_3) = \psi(j_1 m_1)\psi(j_2 m_2)\psi(j_3 m_3). \tag{6.2}$$

We may consider a different set of commuting operators, such as

$$\mathbf{J}_1^2, \mathbf{J}_2^2, \mathbf{J}_3^2, \mathbf{J}_{12}^2, \mathbf{J}^2, J_z \tag{6.3}$$

Here

$$\mathbf{J}_{12} = \mathbf{J}_1 + \mathbf{J}_2$$

and

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3. \tag{6.4}$$

We shall call their eigenfunction $\psi(j_1 j_2 j_3 J_{12} JM)$ the wave function of the coupled angular momenta. It may be expressed in terms of the functions (6.2) as follows

$$\psi(j_1 j_2 j_3 J_{12} JM) = \sum_{m_1 m_2 m_3} \psi(j_1 m_1)\psi(j_2 m_2)\psi(j_3 m_3) \times$$

$$\times (j_1 j_2 j_3 m_1 m_2 m_3 | j_1 j_2 j_3 J_{12} JM). \qquad (6.5)$$

The quantities

$$(j_1 j_2 j_3 m_1 m_2 m_3 | j_1 j_2 j_3 J_{12} JM) \qquad (6.6)$$

which enter into the above expansion shall be called the generalized Clebsch-Gordan coefficients for three coupled angular momenta /Levinson 1957a/. The quantum number J_{12} of the operator $\mathbf{J}_{12}^2$ will be called the intermediate angular momentum parameter, or simply the intermediate angular momentum.

Functions of coupled angular momenta can be constructed from the formulas of vector addition of two angular momenta. Using (3.3) twice we have

$$\psi(j_1 j_2 j_3 J_{12} JM) = \sum_{m_1 m_2 m_3 M_{12}} \psi(j_1 m_1)\psi(j_2 m_2)\psi(j_3 m_3) \times$$

$$\times (j_1 j_2 m_1 m_2 | j_1 j_2 J_{12} M_{12})(J_{12} j_3 M_{12} m_3 | J_{12} j_3 JM). \qquad (6.7)$$

Comparing (6.5) and (6.7) we obtain the following expression for the generalized Clebsch-Gordan coefficient

$$(j_1 j_2 j_3 m_1 m_2 m_3 | j_1 j_2 j_3 J_{12} JM) =$$

$$= \sum_{M_{12}} (j_1 j_2 m_1 m_2 | j_1 j_2 J_{12} M_{12})(J_{12} j_3 M_{12} m_3 | J_{12} j_3 JM). \qquad (6.8)$$

The summation above is purely formal, as it follows from the condition of nonvanishing of the Clebsch-Gordan coefficients that the parameter of summation M_{12} can have only the value $m_1 + m_2$

The subscripts of the intermediate angular momentum denote the angular momenta from which it is compounded. The set of subscripts of all intermediate and resultant angular momenta indicates the scheme of coupling of the angular momenta. Thus, the function

$$\psi(j_1 j_2 j_3 j_4 J_{13} J_{24} J_{1324} M_{1324}) \qquad (6.9)$$

is constructed by the coupling $\mathbf{J}_1 + \mathbf{J}_3 = \mathbf{J}_{13}$, $\mathbf{J}_2 + \mathbf{J}_4 = \mathbf{J}_{24}$ and $\mathbf{J}_{13} + \mathbf{J}_{24} = \mathbf{J}$, which we shall abbreviate as

$$A = \big((1+3)+(2+4)\big). \qquad (6.10)$$

Here and in the following the capital letters A, B,... will denote the coupling scheme. In the coupling scheme it is convenient to differentiate between the "distribution" of intermediate couplings, determined by the arrangement of the brackets in the notation (6.10), and the "coupling sequence", determined by the arrangement of the indices of the component angular momenta j_i in the same /Levinson and Vanagas 1957/. For the wave functions of coupled angular momenta it is convenient to use a notation which sets the coupling scheme in evidence. Thus, instead of (6.9)

we write

$$\psi\Big(\big((j_1j_3)J_{13}(j_2j_4)J_{24}\big)J_{1324}M_{1324}\Big). \tag{6.11}$$

In this notation there is no loss of clarity even when the indices are suppressed. Where a concrete specification of the coupling scheme is not required, one can write, instead of (6.11)

$$\psi\big((j_1j_2j_3j_4)^A a_1 a_2 JM\big) \tag{6.12}$$

where a_1 and a_2 are the intermediate angular momenta. In the general case a wave function of the coupled angular momenta is written in a similar fashion

$$\psi\big((j_1 \cdots j_n)^A aJM\big), \tag{6.13}$$

where a is the set of intermediate angular momenta $a_1, a_2, \cdots, a_{n-2}$. It is the eigenfunction of the set of operators

$$\mathbf{J}_1^2, \cdots, \mathbf{J}_n^2, \mathbf{a}_1^2, \cdots, \mathbf{a}_{n-2}^2, \mathbf{J}^2, J_z. \tag{6.14}$$

In terms of the eigenfunctions of the operators

$$\mathbf{J}_1^2, \cdots, \mathbf{J}_n^2, J_{1z}, \cdots, J_{nz} \tag{6.15}$$

(6.13) is expressed, like (6.4), as

$$\psi\big((j_1 \cdots j_n)^A aJM\big) =$$

$$= \sum_{m_1 \cdots m_n} \psi(j_1 m_1) \cdots \psi(j_n m_n)(j_1 m_1 \cdots j_n m_n | (j_1 \cdots j_n)^A aJM\big). \tag{6.16}$$

The coefficients of this transformation are the generalized Clebsch-Gordan coefficients for an arbitrary number of angular momenta and an arbitrary scheme of coupling. In the notation (6.16) for the generalized Clebsch-Gordan coefficient it is natural to write the usual Clebsch-Gordan coefficient as $(j_1 m_1, j_2 m_2 | (j_1 j_2) j m)$, as in Fano /1951/.

It is often convenient to specify the coupling sequence in terms of the permutation

$$P = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \tag{6.17}$$

which must be carried out on the natural sequence $1, 2, \ , n$ to obtain the required sequence. For given P one should understand by A only the distribution, and drop the indices in the writing of A. Then (6.10) will be represented by

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \tag{6.10a}$$

$$A = \big((\cdot + \cdot) + (\cdot + \cdot)\big), \tag{6.10b}$$

where the dots indicate the positions in which the angular momenta are arranged according to the lower row of (6.10a).

For the above characterization of the scheme of addition the eigenfunctions of

the coupled angular momenta will be written as follows

$$\psi\left((Pj_1 \cdots j_n)^A aJM\right). \tag{6.18}$$

It should also be noted that in this and analogous formulas the indices indicate not only the serial numbers of the angular momenta but also the corresponding coordinates. This means that the permutation operator (6.17) also acts on the coordinates. For this reason (6.18), in an amplified form, will appear as

$$\psi\left(\left(Pj_1(1) \quad j_n(n)\right)^A aJM\right) =$$

$$= \psi\left(\left(j_{i_1}(i_1) \cdots j_{i_n}(i_n)\right)^A aJM\right), \tag{6.19}$$

where $i_1, \cdots, i_n$ indicate the coordinates with the same indices.

The indices of the coordinates must be explicitly written when the coupled angular momenta are numerically equal, such as in

$$\psi\left(\left(j_1 \cdots j(i) \cdots j(k) \cdot j_n\right)^A aJM\right). \tag{6.20}$$

7. Group-theoretic considerations on the generalized Clebsch-Gordan coefficients

From section 3 we know that the usual Clebsch-Gordan coefficients are the elements of the matrix which reduces the direct product of two irreducible representations of the three-dimensional rotation group. It is not difficult to see that the generalized Clebsch-Gordan coefficients are elements of the matrix which reduces the direct product of an arbitrary number of irreducible representations of the same group. We shall satisfy ourselves of this by an explicit examination of the direct product.

Let us suppose that we are given the direct product of n irreducible representations

$$D_{j_1} \times D_{j_2} \times \cdots \times D_{j_n}. \tag{7.1}$$

The matrix which reduces it to the 'quasi-diagonal' form shall be denoted by $C_{j_1 \cdots j_n}$. We then have

$$C^{-1}_{j_1 \cdots j_n} \cdot (D_{j_1} \times \cdots \times D_{j_n}) \cdot C_{j_1 \cdots j_n} =$$

$$= D_{j_1 + \cdots + j_n} + \alpha_{j_1 + \cdots + j_n - 1} D_{j_1 + \cdots + j_n - 1} \dot{+}$$

$$\dot{+} \cdots \dot{+} \alpha_J D_J \dot{+} \cdots \tag{7.2}$$

Here α_J denotes the number of submatrices with the same value of J, further

$$\alpha_{j_1 + \cdots + j_n} = 1. \tag{7.3}$$

Clearly, the following relation holds true

$$\sum_J \alpha_J (2J+1) = \prod_{i=1}^{n} (2j_i + 1), \tag{7.4}$$

as the order of the decomposed matrix should equal the order of the one from which we had started.

The reduction indicated in (7.2) is carried out by the successive reduction of the direct products of two irreducible representations. It is made definite by specifying the scheme of reduction, which is the group-theoretic analogue of the scheme of addition of the preceding section. It is therefore denoted by the symbols of the preceding section. Thus, if the scheme of reduction is such that first the direct product $D_{j_1} \times D_{j_2}$ is reduced and then $D_{J_{12}} \times D_{j_3}$, the order is symbolized by the formula $A' = \big((1+2)+3\big)$.

Each reduction of the product of two representations results in the appearance of a usual Clebsch-Gordan coefficient. Owing to this the elements of the matrix $C_{j_1 \ldots j_n}$ are expressed in terms of the elements of the matrix C with two subscripts. In other words, the generalized Clebsch-Gordan coefficients are expressed in terms of the usual ones, as we already saw in the preceding section. For example, let $n = 3$ in (7.2) and the scheme of reduction be A', we then have

$$C^{-1}_{j_1 j_2 j_3} \cdot (D_{j_1} \times D_{j_2} \times D_{j_3}) \cdot C_{j_1 j_2 j_3} =$$

$$= C^{-1}_{J_{12} j_3} \cdot \Big(\Big(C^{-1}_{j_1 j_2} \cdot (D_{j_1} \times D_{j_2}) \cdot C_{j_1 j_2} \Big) \times D_{j_3} \Big) \cdot C_{J_{12} j_3} \cdot \tag{7.5}$$

Using (3.11a), the right-hand side of the above may be expressed in the following form

$$\sum (J_{12} j_3 JM \,|\, J_{12} j_3 M_{12} m_3)(j_1 j_2 J_{12} M_{12} \,|\, j_1 j_2 m_1 m_2) \times$$

$$\times (m_1 m_2 m_3 \,|\, D_{j_1} \times D_{j_2} \times D_{j_3} \,|\, m_1' m_2' m_3') \times$$

$$\times (j_1 j_2 m_1' m_2' \,|\, j_1 j_2 J_{12} M_{12}')(J_{12} j_3 M_{12}' m_3' \,|\, J_{12} j_3 JM), \tag{7.6}$$

where the summation is carried out over $m_1\ m_2\ m_3\ m_1'\ m_2'\ m_3'\ M_{12}\ M_{12}'$. A comparison with (6.8) shows that the elements of the matrix $C_{j_1 j_2 j_3}$ are the generalized Clebsch-Gordan coefficients for the case of three angular momenta and the scheme of reduction A'.

The matrix which reduces the direct product (7.1) is unitary. Consequently, we have the following conditions of orthonormality of the generalized Clebsch-Gordan coefficients

$$\sum_{m_1 \ldots m_n} \Big((j_1 \ldots j_n)^A a JM \,|\, j_1 m_1 \ldots j_n m_n \Big) \times$$

$$\times \Big(j_1 m_1 \ldots j_n m_n \,|\, (j_1 \ldots j_n)^A a' J' M' \Big) =$$

$$= \delta(a, a')\, \delta(J, J')\, \delta(M, M'), \tag{7.7a}$$

and

$$\sum_{aJM}\left(j_1m_1 \quad \cdot j_nm_n \mid (j_1 \cdots j_n)^A aJM\right)\times$$

$$\times\left((j_1 \quad j_n)^A aJM \mid j_1m_1' \quad j_nm_n'\right) =$$

$$= \delta(m_1, m_1) \cdots \delta(m_n, m_n'), \tag{7.7b}$$

where

$$\delta(a, a') = \delta(a_1, a_1') \quad \delta(a_{n-2}, a'_{n-2})$$

Further, the generalized Clebsch-Gordan coefficients satisfy the reality condition

$$\left(j_1m_1 \quad j_nm_n \mid (j_1 \cdots j_n)^A aJM\right) =$$

$$= \left((j_1 \quad j_n)^A aJM \mid j_1m_1 \quad j_nm_n\right), \tag{7.8}$$

which is the direct consequence and generalization of (3.6).

It is easily seen that the generalized Clebsch-Gordan coefficients are nonzero when

$$\sum_{i=1}^{n} j_i \geqslant J \geqslant 2j_k - \sum_{i=1}^{n} j_i, \quad (k = 1, 2, \cdots, n), \tag{7.9a}$$

$$\sum_{i=1}^{n} j_i + J = \text{integer}, \tag{7.9b}$$

$$\sum_{i=1}^{n} m_i = M, \tag{7.9c}$$

which are the generalizations of (3.8). They may be referred to as the polygon conditions.

We also give the following formula

$$(m_1 \mid D_{j_1} \mid m_1') \cdots (m_n \mid D_{j_n} \mid m_n') =$$

$$= \sum_{aJMM'}\left(j_1m_1 \cdots j_nm_n \mid (j_1 \cdots j_n)^A aJM\right)(M \mid D_J \mid M')\times$$

$$\times\left((j_1 \cdots j_n)^A aJM' \mid j_1m_1' \cdots j_nm_n'\right), \tag{7.10}$$

which is the direct generalization of (3.11b).

8. The transformation matrix

Let us examine two systems of wave functions of coupled angular momenta constructed by the two different schemes of addition A and B. We shall represent the passage from one system into the other in the following form

$$\psi\left((j_1 \cdots j_n)^B bJM\right) =$$

$$= \sum_{aJ'M'}\psi\left((j_1 \quad j_n)^A aJ'M'\right)\times$$

$$\times\left((j_1 \quad j_n)^A aJ'M' \mid (j \cdot j_n)^B bJM\right), \tag{8.1}$$

where the second factor in the right-hand side is the matrix which transforms the eigenfunctions of the scheme of addition A to the eigenfunctions of the scheme B. For the sake of brevity, from now on we shall refer to this matrix as the transformation matrix.

Multiplying (8.1) by ψ^* and integrating over all the variables, we obtain the equation

$$\left((j_1 \cdots j_n)^A a J'M' \mid (j_1 \cdots j_n)^B b JM\right) =$$

$$= \delta(J,J')\delta(M,M') \int \psi^*\left((j_1 \cdots j_n)^A a J'M'\right) \times$$

$$\times \psi\left((j_1 \cdots j_n)^B b JM\right) dx, \qquad (8.2)$$

which shows that the transformation matrix is diagonal in J and M, which agrees with (3.1), from the latter it also follows that the elements of the transformation matrix do not depend on M. One may therefore drop M when writing an element of the transformation matrix.

If the sequence of the coupling is given by the permutation (6.17), we can write an element of the transformation matrix as

$$\left((P_1 j_1 \cdots j_n)^A a J \mid (P_2 j_1 \cdots j_n)^B b J\right), \qquad (8.3)$$

where P_1 and P_2 are the permutations which give the coupling sequences in the schemes A and B respectively.

It may easily be seen that in general the equality

$$\left((P_1 j_1 \cdots j_n)^A a J \mid (P_2 j_1 \cdots j_n)^B b J\right) =$$

$$= \left((P_1 P' j_1 \quad j_n)^A a J \mid (P_2 P' j_1 \cdots j_n)^B b J\right) \qquad (8.4)$$

holds true only when P' rearranges the subscripts of numerically equal angular momenta. If all j_i are equal to each other, it follows from (8.4) that

$$\left(\left(P_1 P_1^{-1} j(1) \cdots j(n)\right)^A a J \mid \left(P_2 P_1^{-1} j(1) \quad j(n)\right)^B b J\right) =$$

$$= \left(\left(E j(1) \quad j(n)\right)^A a J \mid \left(P_3 j(1) \quad j(n)\right)^B b J\right), \qquad (8.5)$$

where $P_3 = P_2 P_1^{-1}$ and E is the identity permutation. The above equality means that for the case $j_1 = j_2 = \cdots$ the transformation matrix, as a function of the coupling sequence, may be written in terms of a single permutation.

From (6.16) and (8.2) we obtain

$$\left((j_1 \cdots j_n)^A a J \mid (j_1 \cdots j_n)^B b J\right) =$$

$$= \sum_{m_1 \cdots m_n} \left((j_1 \cdots j_n)^A a J M \mid j_1 m_1 \cdots j_n m_n\right) \times$$

$$\times \left(j_1 m_1 \cdots j_n m_n \mid (j_1 \cdots j_n)^B b J M\right). \qquad (8.6)$$

Taking (7.8) into account we conclude that

$$\left((j_1 \cdots j_n)^A aJ \,|\, (j_1 \cdots j_n)^B bJ\right) =$$

$$= \left((j_1 \cdots j_n)^B bJ \,|\, (j_1 \cdots j_n)^A aJ\right), \tag{8.7}$$

i.e., the transformation matrix is real if the Clebsch-Gordan coefficients are real.

Remembering that the transformation matrix is independent of M, one can write (8.6) as follows

$$\left((j_1 \quad j_n)^A aJ \,|\, (j_1 \cdots j_n)^B bJ\right) =$$

$$= (J)^{-1} \sum_{m_1 \quad m_n M} \left((j_1 \cdots j_n)^A aJM \,|\, j_1 m_1 \quad j_n m_n\right) \times$$

$$\times \left(j_1 m_1 \cdots j_n m_n \,|\, (j_1 \cdots j_n)^B bJM\right). \tag{8.8}$$

For the particular case $A = B$ and $j_1 = j_2 = \cdots = j_n$ we have, according to (8.7),

$$\left(\left(P_1 j(1) \quad j(n)\right)^A aJ \,|\, \left(P_2 j(1) \cdots j(n)\right)^A a'J\right) =$$

$$= \left(\left(P_2 j(1) \cdots j(n)\right)^A a'J \,|\, \left(P_1 j(1) \cdots j(n)\right)^A aJ\right). \tag{8.9}$$

Further, using (8.4) we obtain

$$\left(\left(j(1) \cdots j(n)\right)^A aJ \,|\, \left(P_2 P_1^{-1} j(1) \cdots j(n)\right)^A a'J\right) =$$

$$= \left(\left(j(1) \cdots j(n)\right)^A a'J \,|\, \left(P_1 P_2^{-1} j(1) \cdots j(n)\right)^A aJ\right). \tag{8.10}$$

In view of the fact that

$$P_2 P_1^{-1} = (P_1 P_2^{-1})^{-1} = P \tag{8.11}$$

it follows from (8.10) that the two such matrices given by the permutations P and P^{-1} are equal.

If in the left- and right-hand sides of the matrix there are intermediate angular momenta which differ only in the order of the subscripts, such as J_{1245} and J_{5241}, then the matrix is diagonal in these angular momenta. This follows from the fact that the two intermediate angular momenta are the eigenvalues of the same operator $(\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_4 + \mathbf{J}_5)^2$. The order of indices of these angular momenta is therefore immaterial.

We note that upon permutation of two directly coupled angular momenta (these may be either the coupled angular momenta j_i or the intermediate angular momenta), the transformation matrix is multiplied by a phase factor. Thus

$$\left(\left((j_1 j_2) J_{12} (j_3 j_4) J_{34}\right) J \,\middle|\, \left(\left((j_1 j_3) J_{13} j_2\right) J_{123} j_4\right) J\right) = (-1)^{(J_{12} + J_{34} - J) + (j_1 + j_2 - J_{12})} \times$$

$$\times \left(\left((j_3 j_4) J_{34} (j_2 j_1) J_{12}\right) J \,\middle|\, \left(\left((j_1 j_3) J_{13} j_2\right) J_{123} j_4\right) J\right) \tag{8.12}$$

Here in the left-hand side of the matrix J_{12} and J_{34}, as well as j_1 and j_2, have been permuted. Upon permutation of the angular momenta J_a and J_b there results, in accordance with (4.7c), the additional phase factor $(-1)^{J_a+J_b-J_{ab}}$.

9. Simplification of the transformation matrix

Let us consider the following transformation matrix

$$\left((l_1 \cdots l_k)^A aLj_1 \cdots j_n\right)^B bJ \,|\, \left((l_1 \cdots l_k)^{\bar{A}} \bar{a} \bar{L} j_1 \cdots j_n\right)^{\bar{B}} \bar{b} J\Big). \tag{9.1}$$

Here on the left-hand side the angular momenta $l_1, \ldots, l_k$ are added up to L according to the scheme A, B is then added to $j_1, \ldots, j_n$ according to scheme B. On the right-hand side we have, instead of the schemes A and B, the schemes $\bar{A}$ and $\bar{B}$ respectively. This matrix, as follows from the end of the last section, is diagonal in L. We can regard the transformation of a scheme of coupling of the set $l_1, \ldots, l_k$ as a transformation of the eigenfunctions of the operator $\mathbf{L}^2$, which does not depend on the scheme of coupling of the angular momenta $Lj_1 \ldots j_n$. Consequently, in accordance with (3.4), (9.1) can be factorized into

$$\delta(L, \bar{L})\left((l_1 \cdots l_k)^A aL \,|\, (l_1 \cdots l_k)^{\bar{A}} \bar{a} \bar{L}\right) \times$$

$$\times \left((Lj_1 \cdots j_n)^B bJ \,|\, (Lj_1 \cdots j_n)^{\bar{B}} \bar{b} J\right). \tag{9.2}$$

The first factor corresponds to the transformation of the set of angular momenta $l_1 \ldots l_k$, and the second factor to the transformation of the coupling of the resultant angular momentum of this set to other angular momenta. If one of these transformations is the identity, the corresponding matrix element degenerates into a product of Kronecker deltas. If the set $l_1 \cdots l_k$ is transformed identically, i.e., $A = \bar{A}$, then

$$\left((l_1 \cdots l_k)^A aLj_1 \cdots j_n\right)^B bJ \,\Big|\, \left((l_1 \cdots l_k)^A a'L'j_1 \cdots j_n\right)^{\bar{B}} \bar{b} J\Big) =$$

$$= \delta(L, L')\,\delta(a, a')\left((Lj_1 \cdots j_n)^B bJ \,\Big|\, (Lj_1 \cdots j_n)^{\bar{B}} \bar{b} J\right). \tag{9.3}$$

If the scheme of coupling of the angular momentum L to the angular momenta $j_1 \ldots j_n$ is transformed identically, i.e., $B = \bar{B}$, then

$$\left((l_1 \cdots l_k)^A aLj_1 \cdots j_n\right)^B bJ \,\Big|\, \left((l_1 \cdots l_k)^{\bar{A}} \bar{a} \bar{L} j_1 \quad j_n\right)^B b'J\Big) =$$

$$= \delta(L, \bar{L})\,\delta(b, b')\left((l_1 \cdots l_k)^A aL \,\Big|\, (l_1 \cdots l_k)^{\bar{A}} \bar{a} L\right). \tag{9.4}$$

Let us consider, for instance, the matrix

$$\Big|\Big(\Big(\Big((j_1 j_2)J_{12} j_3\Big)J_{123} j_4\Big)J_{1234}(j_5 j_6)J_{56}\Big)J\Big|$$

$$\Big|\Big(\Big(j_2(j_1 j_3)J_{12}\Big)J_{213}\Big((j_4 j_6)J_{46} j_5\Big)J_{465}\Big)J\Big). \qquad (9.5)$$

This matrix produces a transformation which consists of changing the scheme of coupling within the set j_1, j_2, j_3 and of changing the scheme of coupling of the resultant angular momentum of this set to the angular momenta j_4, j_5 and j_6. On the basis of (9.2) the matrix element (9.5) therefore becomes

$$\delta(J_{123}, J_{213})\Big(\Big((j_1 j_2)J_{12} j_3\Big)J_{123}\Big|\Big(j_2(j_1 j_3)J_{13}\Big)J_{123}\Big)\times$$

$$\times\Big(\Big((J_{123} j_4)J_{1234}(j_5 j_6)J_{56}\Big)J\Big|\Big(J_{123}\Big((j_4 j_6)J_{46} j_5\Big)J_{465}\Big)J\Big) \qquad (9.6)$$

We further note that if one of the component angular momenta is zero, it may be dropped when writing the transformation matrix. Thus, for $j_2 = 0$

$$\Big(\Big((j_1 j_2)J_{12}(j_3 j_4)J_{34}\Big)J\Big|\Big((j_1 j_3)J_{13}(j_2 j_4)J_{24}\Big)J\Big) =$$

$$= \delta(J_{12}, j_1)\delta(J_{24}, j_4)\Big(\Big(j_1(j_3 j_4)J_{34}\Big)J\Big|\Big((j_1 j_3)J_{13} j_4\Big)J\Big). \qquad (9.7)$$

This equality follows from (4.9a).

Let us consider two further cases in which the transformation is susceptible of simplification. In the first case the transformation only removes the one angular momentum j_0 from the set $l_1, \cdots, l_k, j_0$. This matrix may be simplified as follows

$$\Big(\Big((l_1 \ldots l_k j_0)^A a J_0 j_1 \ldots j_n\Big)^B b J\Big|\Big((l_1 \ldots l_k)^C c L j_0 j_1 \ldots j_n\Big)^D d J\Big) =$$

$$= \Big((l_1 \ldots l_k j_0)^A a J_0\Big|\Big((l_1 \ldots l_k)^C c L J_0\Big)J_0\Big)\times$$

$$\times\Big(\Big((L J_0)J_0 j_1 \ldots j_n\Big)^B b J\Big|(L J_0 j_1 \ldots j_n)^D d J\Big). \qquad (9.8)$$

To be convinced of the validity of this equation one need only write the matrix in the following form

$$\Big(\ldots\Big|\ldots\Big) = \sum\Big(\ldots\Big|\prod\Big)\Big(\prod\Big|\ldots\Big), \qquad (9.9)$$

with the intermediate coupling of the angular momenta

$$\prod = \Big(\Big((l_1 \ldots l_k)^C c' L' j_0\Big)J_0' j_1 \ldots j_n\Big)^B b' J, \qquad (9.10)$$

and use (9.4) and (9.3) for the matrices under the summation sign. The summation in (9.9) is carried out over the intermediate angular momenta $c'L'J_0b'$.

Let us, for instance, consider the matrix

$$\Big(\Big((j_1j_2)J_{12}\Big(\big(j_3j_4\big)J_{34}j_5\Big)J_{345}\Big)J\Big|$$

$$\Big|\Big((j_1j_3)J_{13}\Big(\big(j_4j_5\big)J_{45}j_2\Big)J_{452}\Big)J\Big). \tag{9.11}$$

The transformation which it produces removes the angular momentum j_3 from the set j_3, j_4, j_5 in the left-hand side and forms the set j_4, j_5 in the right side. From (9.8), the matrix element (9.11) is therefore

$$\Big(\Big(\big(j_3j_4\big)J_{34}j_5\Big)J_{345}\Big|\Big(\big(j_4j_5\big)J_{45}j_3\Big)J_{345}\Big)\times$$

$$\times\Big(\Big((j_1j_2)J_{12}\,(J_{45}j_3)J_{345}\Big)J\Big|\Big((j_1j_3)J_{13}\,(J_{45}j_2)J_{452}\Big)J\Big). \tag{9.12}$$

The transformation matrix may also be simplified when in its left-hand side the angular momenta $s_1, \ldots, s_n$ attach themselves to the coupled set of angular momenta $l_1, \ldots, l_k$, while in the right-hand side, as opposed to it, $l_1, \ldots, l_k$ attach themselves to the coupled set $s_1, \ldots, s_n$. The simplification has the following form

$$\Big(\Big((l_1\ldots l_k)^A aLs_1\ldots s_n\Big)^B bJ\Big|\Big((s_1\ldots s_n)^C cSl_1\ldots l_k\Big)^D dJ\Big)=$$

$$=\Big((Ls_1\ldots s_n)^B bJ\Big|\Big(L(s_1\ldots s_n)^C cS\Big)J\Big)\times$$

$$\times\Big(\Big((l_1\ldots l_k)^A aLS\Big)J\Big|(Sl_1\ldots l_k)^D dJ\Big). \tag{9 13}$$

This equation may be established, like (9 8), by choosing an intermediate scheme of the form

$$\prod=\Big((l_1\ldots l_k)^A a'L'(s_1\ldots s_n)^C c'S'\Big)J. \tag{9.14}$$

Consider, for example, the matrix

$$\Big(\Big(\Big(\big((j_1j_2)J_{12}j_3\big)J_{123}j_4\Big)J_{1234}j_5\Big)J\Big|$$

$$\Big|\Big(\Big(\big((j_3j_5)J_{35}j_4\big)J_{345}j_1\Big)J_{1345}j_2\Big)J\Big). \tag{9.15}$$

In this matrix the angular momenta j_3, j_4 and j_5 are attached to the set j_1, j_2 in the left-hand side, and j_1 and j_2 are attached to the set j_3, j_4 and j_5 in the

right-hand side. From (9.13) this matrix is therefore

$$\left(\left(\left((J_{12}j_3)J_{123}j_4\right)J_{1234}j_5\right)J\middle|\left(\left((j_3j_5)J_{35}j_4\right)J_{345}J_{12}\right)J\right)\times$$

$$\times\left(\left((j_1j_2)J_{12}J_{345}\right)J\middle|\left(J_{345}j_1\right)J_{1345}J_2\right)J\right). \qquad (9.16)$$

The formulas obtained for the simplification of the transformation matrices considerably facilitate their calculation and are widely used in deriving various formulas which contain the transformation matrix or similar quantities.

10. Generalized Wigner coefficients and their properties

Just as the usual Clebsch-Gordan coefficients may be expressed in terms of the Wigner coefficients, the generalized Clebsch-Gordan coefficients may be expressed in terms of the generalized Wigner coefficients /Levinson 1957b/

$$\left(j_1m_1\ldots j_nm_n\middle|(j_1\ldots j_n)^Aa_1\ldots a_{n-2}JM\right)=$$

$$=(-1)^{f_A(j_1\ldots j_nJ)}\left[(a_1)\ldots(a_{n-2})\right]^{\frac{1}{2}}\times$$

$$\times(-1)^{J-M}(J)^{\frac{1}{2}}\begin{pmatrix}j_1\ldots j_n & J\\ m_1\ldots m_n & -M\end{pmatrix}^A_{a_1\ldots a_{n-2}}. \qquad (10.1)$$

Here f_A is a linear combination of its parameters which depends on the scheme of addition A. The last factor in (10.1) is the generalized Wigner coefficient Setting $J=j_{n+1}$ and replacing $n+1$ by n, the generalized Wigner coefficient may be represented as follows

$$\begin{pmatrix}j_1\ldots j_n\\ m_1\ldots m_n\end{pmatrix}^A_{a_1\ldots a_{n-3}}. \qquad (10.2)$$

The conditions of nonvanishing of the generalized Wigner coefficients follow from the conditions of nonvanishing of the generalized Clebsch-Gordan coefficients (7.9). They are

1. The parameters $j_1, j_2, \ldots, j_n$ must constitute a polygon with an integral perimeter.

2 The sum of the lower parameters m_i must be zero.

In the particular case where $m_1=\ldots=m_n=0$, it follows from (5.8) that the parameters j_i must form a polygon with an even perimeter This is identical with the condition of nonvanishing of the integral of the product of n associated Legendre polynomials /Jucys et al 1955/, a particular case of which is the condition of nonvanishing of the integral of the product of three associated Legendre polynomials /Condon and Shortley 1949/.

In view of the fact that the generalized Clebsch-Gordan coefficients may be

expressed in terms of the corresponding usual coefficients, the generalized Wigner coefficients may be expressed in terms of the usual Wigner coefficients. The form of this expression depends on the scheme of addition A. Thus, for the scheme of addition $A=\big((1+2)+(3+4)\big)$ we have

$$\begin{pmatrix} j_1 & j_2 & j_3 & j_4 & J \\ m_1 & m_2 & m_3 & m_4 & -M \end{pmatrix}^{A}_{J_{12}J_{34}} =$$

$$=\sum_{M_{12}M_{34}}(-1)^{J_{12}-M_{12}+J_{34}-M_{34}}\begin{pmatrix} j_1 & j_2 & J_{12} \\ m_1 & m_2 & -M_{12} \end{pmatrix}\times$$

$$\times\begin{pmatrix} j_3 & j_4 & J_{34} \\ m_3 & m_4 & -M_{34} \end{pmatrix}\begin{pmatrix} J_{12} & J_{34} & J \\ M_{12} & M_{34} & -M \end{pmatrix}, \qquad (10.3)$$

further, in the equation which corresponds to (10.1),

$$f_A(j_1j_2j_3j_4J)=j_1-j_2-j_3+j_4-J. \qquad (10\,4)$$

The permutational symmetries of the generalized Wigner coefficients result from the corresponding symmetries of the usual Wigner coefficients (5 4). This means that permutation of the columns is possible only within individual Wigner coefficients. The symmetry property related to change in the signs of the projections m_i is of the form

$$\begin{pmatrix} j_1 \cdots j_n \\ m_1 \ldots m_n \end{pmatrix}^{A}_{a_1 \cdots a_{n-3}} =$$

$$=(-1)^{j_1+\cdots+j_n}\begin{pmatrix} j_1 \cdots j_n \\ -m_1 \ldots -m_n \end{pmatrix}^{A}_{a_1 \cdots a_{n-3}}. \qquad (10.5)$$

To prove this one must apply (5.5) to the usual Wigner coefficients in the expression of the generalized Wigner coefficients in terms of the former, change the signs of the projections of all the intermediate angular momenta, and remember that $(-1)^{j+m}=$ $=(-1)^{2j}(-1)^{j-m}$.

The generalized Wigner coefficients are more symmetrical than the generalized Clebsch-Gordan coefficients, owing to the fact that the entire series of generalized Clebsch-Gordan coefficients is expressed in terms of a single generalized Wigner coefficient. Thus, the generalized Clebsch-Gordan coefficients for the schemes of addition $\Big(\big((1+2)+3\big)+4\Big)$ and $\big((1+2)+(3+4)\big)$ are expressed in terms of the same generalized Wigner coefficient

$$\Big(j_1m_1j_2m_2j_3m_3j_4m_4\Big|\Big(\big((j_1j_2)J_{12}j_3\big)J_{123}j_4\Big)JM\Big)=$$

$$=(-1)^{-j_1+j_2-j_4+J_{12}+J_{123}+J}\Big[(J_{12})(J_{123})(J)\Big]^{\frac{1}{2}}\times$$

$$\times(-1)^{J-M}\begin{pmatrix} j_1 & j_2 & j_4 & J & j_3 \\ m_1 & m_2 & m_4 & -M & m_3 \end{pmatrix}_{J_{12}J_{123}} \qquad (10\,6)$$

and

$$\left(j_1 m_1 j_2 m_2 j_3 m_3 j_4 m_4 \middle| \left((j_1 j_2) J_{12} (j_3 j_4) J_{34}\right) JM\right) =$$

$$= (-1)^{j_1 - j_2 - j_3 + j_4 - J} \left[(J_{12})(J_{34})(J)\right]^{\frac{1}{2}} \times$$

$$\times (-1)^{J-M} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 & J \\ m_1 & m_2 & m_3 & m_4 & -M \end{pmatrix}_{J_{12} J_{34}}. \tag{10.7}$$

The orthogonality properties (7.7) of the generalized Clebsch-Gordan coefficients give us the following orthogonality properties of the generalized Wigner coefficients

$$\sum_{\substack{m_i \\ i \neq k}} (-1)^{\sum_{i \neq k} (j_i - m_i)} \begin{pmatrix} j_1 & \cdots & j_k & \cdots & j_n \\ m_1 & \cdots & m_k & \cdots & m_n \end{pmatrix}^A_{a_1 \cdots a_{n-3}} \times$$

$$\times \begin{pmatrix} j_1 & \cdots & j'_k & \cdots & j_n \\ -m_1 & \cdots & -m'_k & \cdots & -m_n \end{pmatrix}^A_{a'_1 \cdots a'_{n-3}} =$$

$$= \delta(j_k, j'_k)(j_k)^{-1} \cdot \delta(m_k, m'_k)(-1)^{j_k - m_k} \times$$

$$\times \prod_{s=1}^{n-3} \delta(a_s, a'_s)(a_s)^{-1}, \tag{10.8a}$$

$$\sum_{j_k m_k a_s} (j_k)(a_1) \ldots (a_{n-3})(-1)^{j_k - m_k} \times$$

$$\times \begin{pmatrix} j_1 & \cdots & j_k & \cdots & j_n \\ m_1 & \cdots & m_k & \cdots & m_n \end{pmatrix}^A_{a_1 \cdots a_{n-3}} \times$$

$$\times \begin{pmatrix} j_1 & \cdots & j_k & \cdots & j_n \\ -m'_1 & \cdots & -m_k & \cdots & -m'_n \end{pmatrix}^A_{a_1 \cdots a_{n-3}} =$$

$$= \prod_{i \neq k} (-1)^{j_i - m_i} \delta(m_i, m'_i). \tag{10.8b}$$

If the first equation is also summed over m_k, putting $j'_k = j_k$ and $m'_k = m_k$ we obtain

$$(a_1) \ldots (a_{n-3}) \sum_{m_i} (-1)^{\sum_i (j_i - m_i)} \begin{pmatrix} j_1 & \cdots & j_n \\ m_1 & \cdots & m_n \end{pmatrix}^A_{a_1 \cdots a_{n-3}} \times$$

$$\times \begin{pmatrix} j_1 & \cdots & j_n \\ -m_1 & \cdots & -m_n \end{pmatrix}^A_{a_1 \cdots a_{n-3}} = 1, \tag{10.9}$$

where the equalities $a'_s = a_s$ $(s = 1, \ldots, n-3)$ are assumed.

Multiplying (7.10) by the matrix element of the representation, integrating over the group and using (2.8), we obtain the formula

$$\int \left(m_1 \left| D_{j_1}(g) \right| m_1'\right) \ldots \left(m_n \left| D_{j_n}(g) \right| m_n'\right) \frac{dg}{G} =$$

$$= \sum_{a_1 \ldots a_{n-3}} (a_1) \ldots (a_{n-3}) \begin{pmatrix} j_1 & \ldots & j_n \\ m_1 & \ldots & m_n \end{pmatrix}^A_{a_1 \ldots a_{n-3}} \times$$

$$\times \begin{pmatrix} j_1 & \ldots & j_n \\ m_1' & \ldots & m_n' \end{pmatrix}^A_{a_1 \ldots a_{n-3}}, \qquad (10.10)$$

which will be required later on (cf. section 13).

CHAPTER III

GRAPHICAL METHODS FOR OPERATIONS WITH SUMS OF PRODUCTS OF WIGNER COEFFICIENTS

In the solution of many problems there appear the sums of products of Clebsch-Gordan coefficients. It is convenient to replace them by the more symmetrical Wigner coefficients and work with sums of products of the latter. Like the Clebsch-Gordan coefficients, these coefficients represent the vector addition of two angular momenta. This suggests the possibility of a graphical representation of expressions containing Wigner coefficients. It might seem that the key geometrical figures in these graphical representations should be triangles representing the individual Wigner coefficients, as vector addition is geometrically represented by a triangle. However, this representation is inconvenient and has found no serious practical application.

It was found that a convenient representation of the Wigner coefficient is a point at which three lines corresponding to the three angular momenta converge. This is a satisfactory starting-point for devising graphical methods of operations with complicated expressions containing Wigner coefficients.

The graphical representation of triads by points was used in the works of Ord-Smith /1954/ and Edmonds /1957/ for quantities which are independent of the projections (in which the projections have already been summed over). However, it was used only for this purpose and not in connection with sums of products of Wigner coefficients. In the works of Levinson /1957a, b, c/ a method was devised for replacing all algebraic calculations with Wigner coefficients by more convenient and general graphical transformations. This method is discussed in general terms in the present chapter.

We discuss the sums of products of Wigner coefficients in section 11 and their graphical representation in the next section. Section 13 is devoted to the expansion of these sums in generalized Wigner coefficients, and section 14 to the transformation of sums of products of Wigner coefficients. Finally, section 15 gives rules for the evaluation of expressions containing the sums of products of Wigner coefficients over angular momentum parameters.

11. Sums of products of Wigner coefficients (jm-coefficients)

We shall examine the products of Wigner coefficients which are summed over twice-repeated projections. We shall demand that the sum satisfy the following conditions

a) The twice-repeated index of summation m should occur once with the positive sign and once with the negative sign, and in both cases be the projection of the same angular momentum j,

b) whenever the projection m of the angular momentum j is to be summed over, the summand should contain the multiplying factor $(-1)^{j-m}$.

The following is an example of a sum which satisfies these requirements

$$\sum_{n_1 n_2 n_3 q_1 q_2} (-1)^{l_1-n_1+l_2-n_2+l-n+k_1-q_1+k_2-q_2} \begin{pmatrix} j_1 & l_1 & k_1 \\ m_1 & n_1 & q_1 \end{pmatrix} \times$$
$$\times \begin{pmatrix} j_2 & l_2 & k_2 \\ -m_2 & n_2 & -q_2 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ -n_1 & -n_2 & n \end{pmatrix} \begin{pmatrix} k_1 & k_2 & l \\ -q_1 & q_2 & -n \end{pmatrix}. \qquad (11.1)$$

The sums encountered in calculations may always be reduced to the form under consideration. Condition (a) may be satisfied by using the rule for the simultaneous change of sign of the magnetic quantum numbers in the Wigner coefficient, as given by (5.5). The phase factor required by (b) may be introduced under the summation sign from the equality $(-1)^{j+m} = (-1)^{2j} \cdot (-1)^{j-m}$ and by using the fact that the sum of the three magnetic quantum numbers must vanish.

The sums of products of the Wigner coefficients may conveniently be called jm-coefficients, as they depend both on the angular momenta j and on those projections m which are not summed over. One such quantity is the expression (11.1), which depends on the angular momenta l_1, l_2, l, k_1, k_2 (and is summed over the projections of these), and on the angular momenta j_1, j_2 and their projections m_1, m_2 which are not summed over. It is convenient to write this jm-coefficient as follows

$$A\left(l,\ l_1 l_2,\ k_1 k_2;\ \begin{matrix} j_1 & j_2 \\ m_1 & -m_2 \end{matrix}\right). \qquad (11.2)$$

The angular momenta whose projections are summed over* may conveniently be grouped into sets. Various criteria may be adopted for this grouping. Thus in (11.2) they are grouped in such a way that the first l forms a triad with each of the following two pairs ($(l\, l_1 l_2)$ and $(l\, k_1 k_2)$ respectively), and in the three pairs $(l_1 l_2)$, $(k_1 k_2)$, $(j_1 j_2)$ the first members and the second members taken together, that is $(l_1\, k_1\, j_1)$ and $(l_2 k_2 j_2)$, also form triads. However, one cannot write different sums in the same form, as they differ essentially from each other. Therefore for each case one must choose a distribution of parameters which will allow a compact notation and will represent the triad structures as clearly as possible. Thus if we take the sum

$$\sum_{n_1 n_2 n_3 n_4} (-1)^{\sum_{i=1}^{4} (l_i - n_i)} \begin{pmatrix} l_1 & j_1 & l_2 \\ -n_1 & m_1 & n_2 \end{pmatrix} \times$$
$$\times \begin{pmatrix} l_2 & j_2 & l_3 \\ -n_2 & m_2 & n_3 \end{pmatrix} \begin{pmatrix} l_3 & j_3 & l_4 \\ -n_3 & m_3 & n_4 \end{pmatrix} \begin{pmatrix} l_4 & j_4 & l_1 \\ -n_4 & m_4 & n_1 \end{pmatrix}, \qquad (11.3)$$

---

* Henceforth we shall call them "contracted" angular momenta, and those with non-summed projections "free" angular momenta—Translator.

a clear way of writing it would be

$$F\begin{pmatrix} l_1 & l_2 & l_3 & l_4 \\ j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}, \tag{11.4}$$

since from this the distribution of angular momenta in triads can be easily seen. However, for compactness it may be written as

$$F\left(l_1 l_2 l_3 l_4;\ \begin{matrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{matrix}\right). \tag{11.5}$$

It would be obvious that the generalized Wigner coefficients are also jm-coefficients. They represent those jm-coefficients which, for a given number of free angular momenta, have a minimum number of contracted angular momenta. The notation for the generalized Wigner coefficients is given in section 10.

It should be noted that in the particular case where all magnetic quantum numbers are summed over, the resulting expression is independent of these. These quantities are called j-coefficients and are studied in the next chapter.

The form of summation which we have chosen for products of Wigner coefficients is closely related to the invariance of the expression

$$\sum_m (-1)^{j-m}\psi(j-m)\,\psi(jm) =$$
$$= (j)^{\frac{1}{2}}\psi\big((jj)J=0,\ M=0\big) \tag{11.6}$$

This is easily seen by introducing (4.9b) into (3.3). Using (4.9b) and (6.5) one can easily verify that

$$\sum_{m_1 m_2 m_3}\psi(j_1 m_1)\,\psi(j_2 m_2)\,\psi(j_3 m_3)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =$$
$$= (-1)^{j_1-j_2+j_3}\psi\big((j_1 j_2 j_3)J=0,\ M=0\big), \tag{11.7}$$

i.e., is also invariant. Equation (11.6) shows that when the Wigner coefficients are contracted the role of the Kronecker delta is taken over by the quantity

$$\delta_j(-m,\ m') = (-1)^{j-m}\delta(m,\ m'). \tag{11.8}$$

This is also clear from the following

$$\sum_m (-1)^{j-m}\begin{pmatrix} j & k & l \\ m & q & n \end{pmatrix}\begin{pmatrix} j & k' & l' \\ -m & q' & n' \end{pmatrix} =$$
$$= \sum_{mm'} (-1)^{j-m}(-1)^{j-m'}\begin{pmatrix} j & k & l \\ m & q & n \end{pmatrix} \times$$
$$\times\begin{pmatrix} j & k' & l' \\ -m' & q' & n' \end{pmatrix}\delta_j(-m,\ m') \tag{11.9}$$

and

$$\sum_{j} T^{jkl}\, U_{j}^{k'l'} = \sum_{jj'} T^{jkl}\, U_{j'}^{k'l'}\, \delta_{j}^{j'}. \tag{11.10}$$

From (11.7) it may be seen that the Wigner coefficient is the analogue of the fully antisymmetric tensor ε_{ikl}, with which one forms the scalar triple product of three vectors. For the sake of generality it is convenient to consider sums containing, in addition to Wigner coefficients, the quantities

$$\begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix} \equiv \begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0 \end{pmatrix} =$$

$$= (j_1)^{-\frac{1}{2}}\, \delta(j_1, j_2) \cdot (-1)^{j_1 - m_1}\, \delta(m_1, -m_2) \tag{11.11}$$

and

$$\begin{pmatrix} j_1 \\ m_1 \end{pmatrix} \equiv \begin{pmatrix} j_1 & 0 & 0 \\ m_1 & 0 & 0 \end{pmatrix} = \delta(j_1, 0)\ \delta(m_1, 0), \tag{11.12}$$

as well as the delta $\delta_j(-m, m')$, although these sums can be simplified in an elementary manner owing to the presence of the Kronecker deltas in (11.8), (11.11) and (11.12).

It will be useful to remember that

$$\begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix} = (-1)^{j_1 + j_2} \begin{pmatrix} j_2 & j_1 \\ m_2 & m_1 \end{pmatrix} \tag{11.13}$$

analogous to (5.4). We note that the triangular delta may also be regarded as a j-coefficient, as from (5.6a) it follows that

$$\{j_1 j_2 j_3\} = \sum_{m_1 m_2 m_3} (-1)^{j_1 - m_1 + j_2 - m_2 + j_3 - m_3} \times$$

$$\times \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{11.14}$$

12. Graphical representation of jm-coefficients

When dealing with the sums of products of Wigner coefficients it is convenient to make use of diagrammatic representations. The elements of these diagrams should

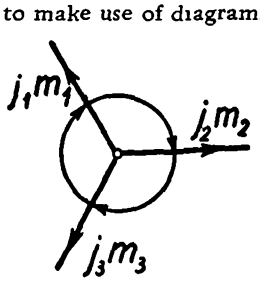

Figure 12.1

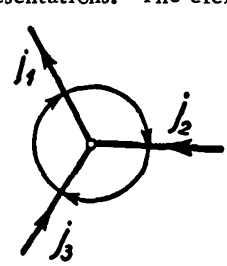

Figure 12.2

Figure 12.3

be diagrams of Wigner coefficients. The Wigner coefficient $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is represented by an oriented node which links three directed lines. These lines (Figure 12.1) correspond to j_1, j_2 and j_3 and their free ends to the projections m_1, m_2 and m_3. By the orientation of the node is meant the cyclic order of the lines j_1, j_2, j_3, , which is shown in the diagram by circular arrows. The direction of the line j_i indicates the sign of the magnetic quantum number m_i in the Wigner coefficient. A line directed away from the node corresponds to the positive sign of the magnetic quantum number, and a line directed towards the node to the negative sign. Thus the Wigner coefficient $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & -m_2 & -m_3 \end{pmatrix}$ may be represented by the diagram of Figure 12.2. Apart from rare exceptions, the magnetic quantum numbers will not be indicated in the diagrams. We shall adopt the convention that the projections of the angular momenta j, l, k and a will be respectively m, n, q and μ, with corresponding indices.

The quantities $\begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix}$ and $\begin{pmatrix} j_1 \\ m_1 \end{pmatrix}$ may be represented analogously. Their diagrams are shown in Figures 12 3 and 12.4 respectively. In Figure 12.3 the circular arrow indicates the order of succession of the angular momenta j_1 and j_2. The signs of m_1 and m_2 are indicated in the same way as above.

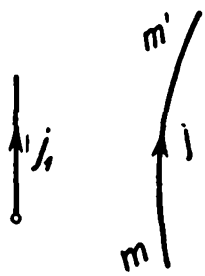

Figure 12.4 Figure 12.5

Figure 12 5 shows the diagrammatic representation of $\delta_j(-m, m')$. This diagram is the directed line of the angular momentum j, the ends of which correspond to the projections m and m'. The signs of m and m' in δ_j are again indicated as above. The direction of the line j may be changed, which corresponds to the symmetry of δ_j in m and m'.

In practice one rarely encounters the representations of the quantities $\begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix}$ and $\begin{pmatrix} j_1 \\ m_1 \end{pmatrix}$. It is therefore more convenient to denote the orientation of the node representing the Wigner coefficient by a sign. A clockwise orientation is denoted by a – sign and a counter-clockwise orientation by a + sign (Figure 12.6).

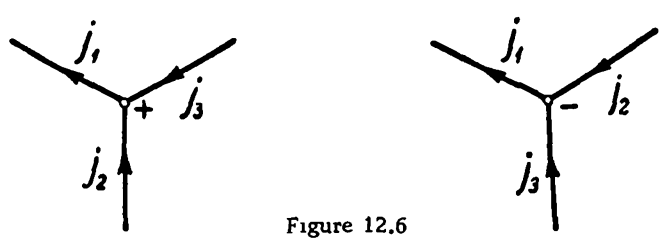

Figure 12.6

Rotation of the diagram does not change the cyclic order of the lines. Owing to the symmetry (5.4) a Wigner coefficient remains unchanged by a cyclic permutation of the columns. Therefore a rotated diagram represents the same Wigner

coefficient as the initial diagram. The angles between the lines as well as their lengths have no significance. Consequently, geometrical deformations of a diagram which preserve the orientation of the node do not change the Wigner coefficient represented by the diagram. Deformations which change the cyclic order of the lines around the node imply a change in the orientation of the node. In accordance with (5.4), the Wigner coefficient represented by such a deformed diagram will differ from that represented by the original diagram by the factor $(-1)^{j_1+j_2+j_3}$. In many cases such a change in the orientation of the node may be compensated by a change in the sign of the node, as changing the sign amounts to the same thing as changing the orientation.

Summation over a magnetic quantum number m is graphically represented by joining the free ends of the corresponding lines*. As an example, let us consider

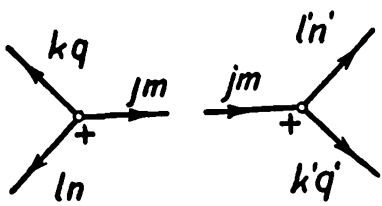

Figure 12.7

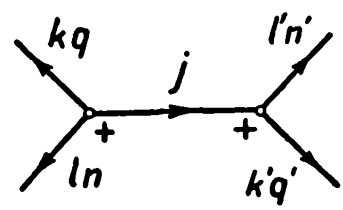

Figure 12.8

the sum in the left-hand side of (11.9). Figure 12.7 shows the diagrams of the Wigner coefficients which appear in the sum. Joining the free ends which correspond to the magnetic quantum number m we obtain the diagram shown in Figure 12.8. This diagram represents the sum under consideration. Let us now consider the sum in the right-hand side of (11.9). In this case we have to represent the two Wigner coefficients and the delta, and join the free ends m, m and m', m' (Figure 12.9). The diagram thus obtained is identical with Figure 12.8.

The diagrams A and F similarly obtained for (11.1) and (11.3) are shown in Figures 12.10 and 12.11 respectively.

---

* The remainder of the book leans heavily on the diagrammatic representations. For a description of the diagrams and of operations with them we introduce the following systematic nomenclature

1. A line with a free end will be called a "free" line.
2. A diagram which contains free lines will be called an "open diagram", . diagram which contains no free lines will be called a "closed" diagram.
3. If two open diagrams are such that each contains a free line referring to the same angular momentum j_i, these two free lines will be called "corresponding" free lines.
4. Joining the free ends of two lines referring to the same angular momentum will be called "contracting" the lines. Two lines may be contracted only when they are similarly directed. This can always be arranged, as it involves only a change in the phase factor
5. Contracting all pairs of corresponding free lines in two open diagrams will be called "contracting" the two diagrams.

Translator.

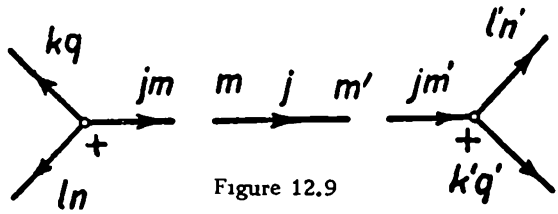

Figure 12.9

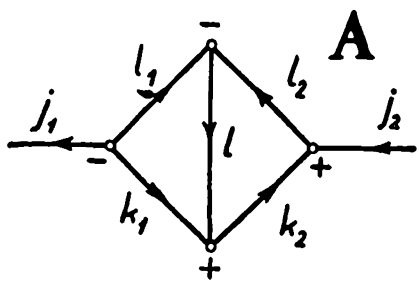

Figure 12.10

Thus, any sum of products of Wigner coefficients, as well as the quantities (11.1), (11.12) and the deltas, can be represented by diagrams consisting of oriented nodes and the directed lines which join them. One, two or three lines may issue from each node. In certain cases the diagram may not contain any node. Thus the diagram in Figure 12.12 represents the j-coefficient

$$\sum_m \delta_j(-m, m)\ (-1)^{j-m} = (j). \qquad (12.1)$$

If the sum contains only Wigner coefficients, three lines must issue from each node in the diagram.

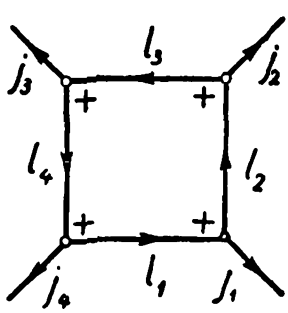

Figure 12.11

Once again, a deformation of the diagram which does not change the orientations of the nodes and lines does not affect the jm-coefficient represented by the diagram. A change in the orientation of a node which joins the lines j_1, j_2, j_3 will multiply the jm-coefficient by $(-1)^{j_1+j_2+j_3}$. Similarly, a change in the cyclic order of the lines j_1 and j_2 which join at a node multiplies the jm-coefficient by $(-1)^{j_1+j_2}$ (from (11.13)). A change in the direction of the line j means the replacement of m by $-m$. Whenever two nodes are joined by the line j, m is an index of summation and a change in its sign multiplies the jm-coefficient by $(-1)^{2j}$.

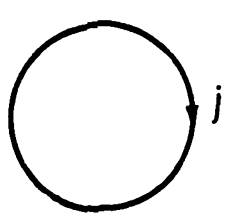

Figure 12.12

Diagrams representing the generalized Wigner coefficients can be constructed only when the scheme of addition is explicitly given. Figure 12.13 shows the generalized

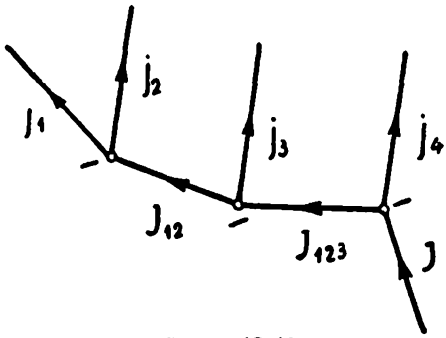

Figure 12.13

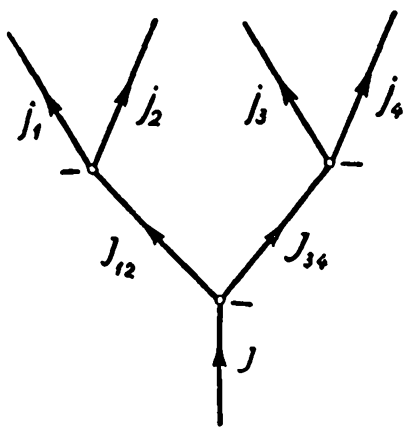

Figure 12.14

Wigner coefficient for the scheme of addition $\Big(\big((1+2)+3\big)+4\Big)$, and Figure 12.4 the same for the scheme $\Big((1+2)+(3+4)\Big)$. Comparing these diagrams we see that these generalized Wigner coefficients are essentially similar. Therefore the generalized Clebsch-Gordan coefficients for the schemes of addition (10.6) and (10.7) are expressed in terms of the same generalized Wigner coefficient.

For any scheme of addition whatsoever, the diagrams of the generalized Wigner coefficients do not contain closed figures. This follows from the fact that the corresponding sums contain the minimum number of contracted angular momenta. It is convenient to give a branching form to the generalized Wigner coefficients, as is done in Figures 12 13 and 12.14. Obviously, any diagram which does not contain closed figures and which is represented in branching form may be related to some generalized Wigner coefficient.

In many cases the structure of a part of the diagram and not of the whole is of importance. In these cases the remaining parts of the diagram may be regarded as arbitrary and replaced by one or several blocks. Thus, if we wish to separate the line j which joins two nodes in a diagram, we may do so by means of the block diagram W in Figure 12.15. The block α replaces all other lines in the diagram together with the scheme of joining

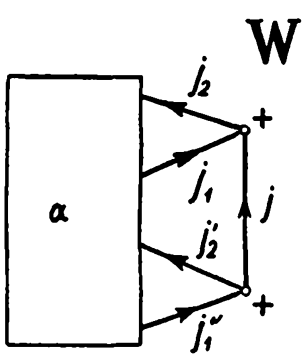

Figure 12 15

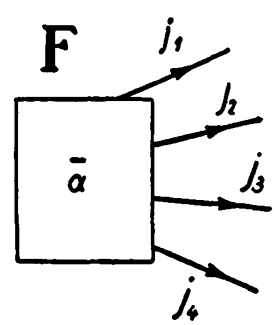

Figure 12.16

these lines to the lines j_1, j_2, j_1' and j_2'. The lines in the block α may include free lines as well as those which join two nodes. If the block consists only of lines of the second kind it is referred to as 'closed'. In general the block will be 'open'. The block diagram in Figure 12.16 indicates that the diagram F contains in all four free lines j_1, j_2, j_1', and j_2'. The remaining lines each join two nodes and the set of these lines is replaced by the closed block $\bar{\alpha}$ (the bar above indicates a closed block). The blocks are usually separated when they are not important for further transformations with diagrams or jm-coefficients. We shall therefore denote them by subscripts in the jm-coefficients. Thus the jm-coefficients corresponding to the diagrams W and F may be written as follows

$$W_\alpha(j_1 j_2; j_1' j_2'; j) \tag{12.2}$$

and

$$F_{\bar{\alpha}}\begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}. \tag{12.3}$$

A block diagram may also contain several blocks.

When constructing the diagram for a jm-coefficient one chooses the simplest and most explicit form, since every change (deformation) of a diagram which does not change the orientations of its nodes and the directions of its lines will not affect the jm-coefficient represented by the diagram.

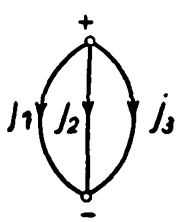

Figure 12.17

We note that diagrams containing nodes with two lines can easily be simplified. Comparing the diagrams (12.3) and (12.5) with the corresponding quantities (11.11) and (11.8) we see that if a line j_1 issues from a node and a line j_2 converges to it, and if the orientation of the node is from the former to the latter, then the node may be dropped. This gives the factor $\delta(j_1, j_2)\,(j_1)^{-\frac{1}{2}}$. If a parameter j is zero, the corresponding line is simply erased.

The triangular delta, as may be seen from (11.14), is represented by the diagram of Figure 12.17.

13. Expansion of jm-coefficients in generalized Wigner coefficients

If a jm-coefficient is not a generalized Wigner coefficient (i.e., if the number of contracted angular momenta is not minimal), it may then be expanded in generalized Wigner coefficients.

Let us consider an arbitrary jm-coefficient, the diagram of which contains the free lines $j_1, j_2, \ldots, j_n$ to which correspond the projections $m_1, m_2, \ldots, m_n$. All the remaining lines $l_1, l_2, \ldots, l_k$ join nodes. We shall replace the set of these by the closed block $\bar{\alpha}$ and denote the jm-coefficient under consideration by $F_{\bar{\alpha}}\begin{pmatrix} j_1 & \cdots & j_n \\ m_1 & \ldots & m_n \end{pmatrix}$. Let us now express F as the sum of products of Wigner coefficients and substitute an equivalent expression for each of the latter in accordance with (5.10). The matrix elements of the representations corresponding to the angular momenta l_t

can be left out. To prove this, consider an angular momentum which appears twice, once with the projection n and once with the projection $-n$. The part of the sum which contains this parameter is therefore

$$\sum_{n}(-1)^{l-n}\begin{pmatrix}\dots l\\ \dots n\end{pmatrix}\begin{pmatrix}\dots & l\\ \dots & -n\end{pmatrix}, \tag{13.1}$$

where the dots indicate other parameters which are of no interest for the present. After introducing (5.10) we will have, instead of (13.1), the expression

$$\sum_{nn'n''}(-1)^{l-n}\begin{pmatrix}\dots l\\ \dots n'\end{pmatrix}\begin{pmatrix}\dots l\\ \dots n''\end{pmatrix}\times$$

$$\times\left(n'\mid D_l(g)\mid n\right)\left(n''\mid D_l(g)\mid -n\right). \tag{13.2}$$

The summation over n is carried out with the aid of the unitarity relation (2.5a) of the matrix D_l, the resulting expression is

$$\sum_{n'n''}\begin{pmatrix}\dots l\\ \dots n'\end{pmatrix}\begin{pmatrix}\dots l\\ \dots n''\end{pmatrix}(-1)^{l-n'}\delta(-n',\ n''). \tag{13.3}$$

This sum is identical with (13.1), thus proving the assertion. Therefore, after introducing (5.10) one may retain only the matrix elements of those representations which correspond to the parameters j_i, i.e.,

$$F_{\bar{\alpha}}\begin{pmatrix}j_1\ \cdots j_n\\ m_1\dots m_n\end{pmatrix}=$$

$$=\sum_{m_1'\dots m_n'}F_{\bar{\alpha}}\begin{pmatrix}j_1\ \cdots j_n\\ m_1'\dots m_n'\end{pmatrix}\left(m_1'\mid D_{j_1}(g)\mid m_1\right)\dots\times$$

$$\times\left(m_n'\mid D_{j_n}(g)\mid m_n\right). \tag{13.4}$$

Let us first consider the general case $n\geqslant 3$. In this case we integrate the above equation over the group, using (10.10). We then obtain the required expansion of the jm-coefficient in generalized Wigner coefficients

$$F_{\bar{\alpha}}\begin{pmatrix}j_1\ \cdots j_n\\ m_1\ \dots\ m_n\end{pmatrix}=$$

$$=\sum_{a_1\cdots a_{n-3}}(a_1)\dots(a_{n-3})\,R_{\bar{\alpha}}(a_1\ \dots\ a_{n-3};\ j_1\ \dots j_n)\times$$

$$\times\begin{pmatrix}j_1\ \cdots j_n\\ m_1\ \dots\ m_n\end{pmatrix}^{A}_{a_1\cdots a_{n-3}}, \tag{13.5}$$

where the expansion coefficient is

$$R_{\bar{\alpha}}(a_1\ \dots\ a_{n-3};\ j_1\ \dots j_n)=$$

$$=\sum_{m_1\cdots m_n}F_{\bar{\alpha}}\begin{pmatrix}j_1\ \cdots j_n\\ m_1\ \dots\ m_n\end{pmatrix}\begin{pmatrix}j_1\ \cdots j_n\\ m_1\ \dots\ m_n\end{pmatrix}^{A}_{a_1\cdots a_{n-3}}. \tag{13.6}$$

In order to reduce the summation over the magnetic quantum numbers in this expression to the adopted standard form, one must make use of (10.5) and of the vanishing of the sum of all the projections in the generalized Wigner coefficient. We then have

$$R_{\bar{x}}(a_1 \ldots a_{n-3};\ j_1 \ldots j_n) =$$

$$= \sum_{m_1 \ldots m_n} (-1)^{j_1 - m_1 + \cdots + j_n - m_n} \times$$

$$\times F_{\bar{a}} \begin{pmatrix} j_1 & \ldots & j_n \\ m_1 & \ldots & m_n \end{pmatrix} \begin{pmatrix} j_1 & \ldots & j_n \\ -m_1 & \ldots & -m_n \end{pmatrix}^{A}_{a_1 \ldots a_{n-3}}. \qquad (13.7)$$

From here it is seen that R is a j-coefficient, the diagram of which may be obtained by contracting the diagram F and the diagram of the generalized Wigner coefficient, retaining for the contracted lines their directions in F. We note that the form of the coefficient R therefore depends on the generalized Wigner coefficient in which the expansion is carried out. It follows from the expansion (13.5) that a jm-coefficient will not vanish only when the free angular momenta satisfy the polygon conditions and when the sum of their projections is zero. Otherwise, the generalized Wigner coefficient appearing in the *right-hand side of* (13.5) will vanish.

It should be noted that for $n = 3$ the summation in (13.5) is trivial. In this case we have

$$F_{\bar{a}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = R_{\bar{a}}(j_1 j_2 j_3) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \qquad (13.8)$$

where

$$R_{\bar{x}}(j_1 j_2 j_3) = \sum_{m_1 m_2 m_3} (-1)^{j_1 - m_1 + j_2 - m_2 + j_3 - m_3} \times$$

$$\times F_{\bar{a}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \qquad (13.9)$$

The diagram of the coefficient R is obtained simply by joining the free lines in diagram F at a single node. The orientation of this node should be identical with that of the node which represents the Wigner coefficient chosen in (13.8).

For $n = 2$ the generalized Wigner coefficient, both in (13.5) and in (13.6), becomes the particular case (5.7) of the usual Wigner coefficient, so that the two parameters j_1 and j_2 must form a "polygon". As a result we obtain

$$F_{\bar{a}} \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix} = R_{\bar{a}}(j_1 j_2) \begin{pmatrix} j_1 & j_2 \\ m_1 & m_2 \end{pmatrix} =$$

$$= \delta(j_1, j_2)\ (j_1)^{-1} (-1)^{j_1 - m_1} \delta(m_1, -m_2)\ R_{\bar{a}}(j_1). \qquad (13.10)$$

The diagram of the j-coefficient $R_{\bar{a}}(j_1 j_2)$ is obtained simply by joining the two free lines in diagram F at one node. The diagram of the j-coefficient $R_{\bar{a}}(j_1)$ is obtained by joining the lines j_1 and j_2 in diagram F to form a single line with the direction of j_1.

For $n = 1$ we have

$$F_{\bar{\alpha}}\begin{pmatrix} j_1 \\ m_1 \end{pmatrix} = R'_{\bar{\alpha}}(j_1)\begin{pmatrix} j_1 \\ m_1 \end{pmatrix} =$$

$$= \delta(j_1, 0)\,\delta(m_1, 0)\,F_{\bar{\alpha}}\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (13.11)$$

The diagram of $R'_{\bar{\alpha}}(j_1)$ is obtained by adding a node to the free end of the line j_1 in diagram $F_{\bar{\alpha}}$.

14. Transformation of jm-coefficients

The formula for the expansion of jm-coefficients in generalized Wigner coefficients which was established in the preceding section allows one to obtain a very general method of expressing jm-coefficients in terms of coefficients with a smaller number of parameters, this method makes possible a clear graphical interpretation.

We shall say that the diagram G is "separable" on n lines $l_1, l_2, \ldots, l_n$ when, if these lines are "cut", the diagram G breaks up into two diagrams A and B, one of which, say B, contains all the free lines in diagram G. Further, in nontrivial cases, diagrams A and B will contain more than one node. Thus diagram G in Figure 14.1 is separable on the four lines l_1, l_2, l_3, l_4, the four lines l_1, s_1, s_2, l_4, and the two lines s_3, s_4. However, it will not be separable on the four lines k_1, k_2, p, s, as both parts then formed will contain free lines from diagram G (the upper part the line j_1 and the lower part the line j_2). If the diagram G is a diagram of a j-coefficient, i.e., does not contain free lines, the condition of separability reduces to the decomposition of the diagram after separation into two parts.

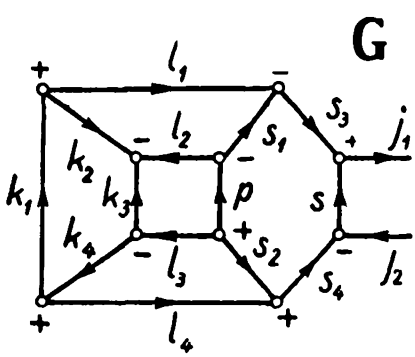

Figure 14 1

In the general case the jm-coefficient G, which is separable on the lines $l_1, l_2, \ldots, l_n$, may be written as the following algebraic expansion.

$$G_{\bar{\alpha}\beta}(l_1 \ldots l_n) = \sum_{n_1 \ldots n_n} (-1)^{l_1 - n_1 + \cdots + l_n - n_n} \times$$

$$\times A_{\bar{\varkappa}}\begin{pmatrix} l_1 \ldots l_n \\ n_1 \ldots n_n \end{pmatrix} B_\beta\begin{pmatrix} l_1 \ldots l_n \\ -n_1 \ldots -n_n \end{pmatrix}. \qquad (14.1)$$

We have initially oriented the line l_i in the direction A to B. The block $\bar{\alpha}$ is closed, as the only free lines in diagram A are the lines $l_1, \ldots, l_n$.

Let us apply formula (13.5) to the jm-coefficient A in (14.1). We obtain the

following equation

$$G^{\bar{\alpha}}_{\beta}(l_1 \ldots l_n) = \sum_{a_1 \ldots a_{n-3}} (a_1) \ldots (a_{n-3})\, R^{\bar{\alpha}}(a_1 \ldots a_{n-3};\ l_1 \ldots l_n) \times$$

$$\times\, C_{\beta}(a_1 \ldots a_{n-3},\ l_1 \ldots l_n), \tag{14.2}$$

where the j-coefficient

$$R^{\bar{\alpha}}(a_1 \ldots a_{n-3};\ l_1 \ldots l_n) = \sum_{n_1 \ldots n_n} (-1)^{l_1 - n_1 + \cdots + l_n - n_n} \times$$

$$\times A^{\bar{\alpha}} \begin{pmatrix} l_1 \ldots l_n \\ n_1 \ldots n_n \end{pmatrix} \begin{pmatrix} l_1 \ldots & l_n \\ -n_1 \ldots & -n_n \end{pmatrix}^{A}_{a_1 \cdots a_{n-3}} \tag{14.3}$$

and the jm-coefficient

$$C_{\beta}(a_1 \ldots a_{n-3};\ l_1 \ldots l_n) = \sum_{n_1 \cdots n_n} (-1)^{l_1 - n_1 + \cdots + l_n - n_n} \times$$

$$\times\, B_{\beta} \begin{pmatrix} l_1 \ldots & l_n \\ -n_1 \ldots & -n_n \end{pmatrix} \begin{pmatrix} l_1 \ldots l_n \\ n_1 \ldots n_n \end{pmatrix}^{A}_{a_1 \cdots a_{n-3}}. \tag{14.4}$$

Thus, formula (14.2) expresses the jm-coefficient G, which is separable on n lines, in the form of an $(n-3)$-fold sum of the product of the j-coefficient R and the jm-coefficient C, each of which contains fewer parameters than the initial jm-coefficient G. From (14.3) and (14.4) it is seen that diagrams R and C are obtained by contracting the respective separated diagrams with the diagram of the chosen generalized Wigner coefficient. By changing the diagram of this coefficient one may choose the most suitable form for the coefficients R and C.

The jm-coefficient G, which is separable on n lines $l_1, l_2, \ldots, l_n$, tends to zero if the parameters $l_1, l_2, \ldots, l_n$ do not satisfy the polygon condition. Clearly, when this condition is not satisfied the generalized Wigner coefficients in (14.3) and (14.4) vanish, and therefore R and C vanish, together with G.

Formula (14.2) is particularly important for $n = 3$, as in this case there is no summation over a_i

$$G^{\bar{\alpha}}_{\beta}(l_1 l_2 l_3) = R^{\bar{\alpha}}(l_1 l_2 l_3)\, C_{\beta}(l_1 l_2 l_3), \tag{14.5}$$

where the j-coefficient

$$R^{\bar{\alpha}}(l_1 l_2 l_3) = \sum_{n_1 n_2 n_3} (-1)^{l_1 - n_1 + l_2 - n_2 + l_3 - n_3} \times$$

$$\times A^{\bar{\alpha}} \begin{pmatrix} l_1 & l_2 & l_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ -n_1 & -n_2 & -n_3 \end{pmatrix}, \tag{14.6}$$

and the jm-coefficient

$$C_{\beta}(l_1 l_2 l_3) = \sum_{n_1 n_2 n_3} (-1)^{l_1 - n_1 + l_2 - n_2 + l_3 - n_3} \times$$

$$\times B_{\beta} \begin{pmatrix} l_1 & l_2 & l_3 \\ -n_1 & -n_2 & -n_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ n_1 & n_2 & n_3 \end{pmatrix}. \tag{14.7}$$

Formula (14.5) shows that a jm-coefficient which is separable on three lines decomposes into the product of a j-coefficient and a jm-coefficient, each of which has a smaller number of parameters. To obtain the diagrams of these coefficients one must, in each part, join the three free lines obtained upon separation at a single node, with the same orientation in the two parts.

A unique case arises for $n=2$. The diagram of the coefficient H, which is separable on two lines (l_1 and l_2), is represented in Figure 14.2. In this case the jm-coefficient H reduces to the product of the j-coefficient R and the jm-coefficient C. Their diagrams are obtained by cutting the lines l_1 and l_2 in diagram H and joining the free lines of each part at one node. As both diagrams contain a node with two lines, they may be simplified, and we obtain

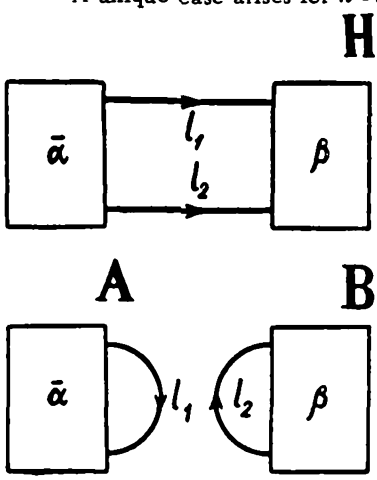

Figure 14.2

$$H_{\bar{\alpha}\beta}(l_1 l_2) = (-1)^{2l_1}(l_1)^{-1}\delta(l_1, l_2)\times \\ \times A_{\bar{\alpha}}(l_1)\ B_{\beta}(l_2). \qquad (14.8)$$

A particular case of this formula is the rule of simplification of the diagrams of jm-coefficients in which two nodes are joined by two lines (Figure 14.3). We first change the direction of the line l_1 and make use of (14.8). Diagram B then reduces to the

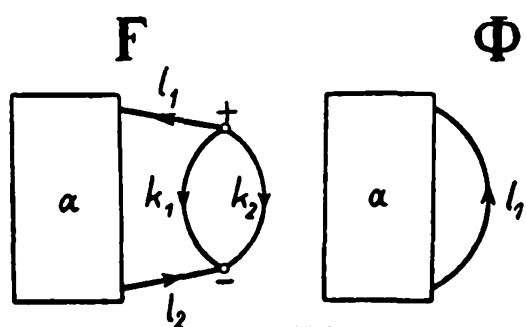

Figure 14.3

diagram of a triangular delta (Figure 12.17). Thus we have

$$F_{\alpha} = \delta(l_1, l_2)\ (l_1)^{-1}\ \{l_1 k_1 k_2\}\ \Phi_{\alpha}. \qquad (14.9)$$

If the diagram is separable on the single line l_1 (Figure 14.2, in which the line l_2 is missing), we obtain analogously

$$H_{\bar{\alpha}\beta}(l_1) = \delta(l_1, 0)\ A_{\bar{\alpha}}(0)\ B_{\beta}(0) \qquad (14.10)$$

Let us use the formula obtained to simplify the diagram in which a line issues and ends on the same node (Figure 14.4). Separating the diagram F on l, erasing l

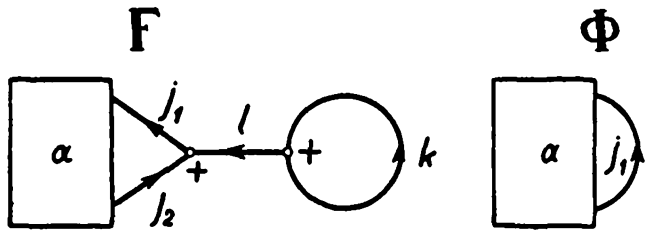

Figure 14.4

in both parts and dropping the nodes with two lines [see end of section 12], we obtain

$$F_\alpha = \delta(l, 0)\ \delta(j_1, j_2)\ (j_1)^{-\frac{1}{2}}\ (k)^{\frac{1}{2}}\ \Phi_\alpha, \qquad (14.11)$$

where Φ_α denotes the jm-coefficient represented in Figure 14.4.

As we have seen, all cases of diagrams separable on $n \leqslant 3$ lines enable the jm-coefficients to be simplified, by expressing them in terms of products of simpler coefficients. For $n = 2$ and 1 one of these coefficients becomes a delta, as we have seen in the particular cases.

To conclude the present section, we give the rule for the simplification of diagrams of jm-coefficients for the case where the value of one of the contracted

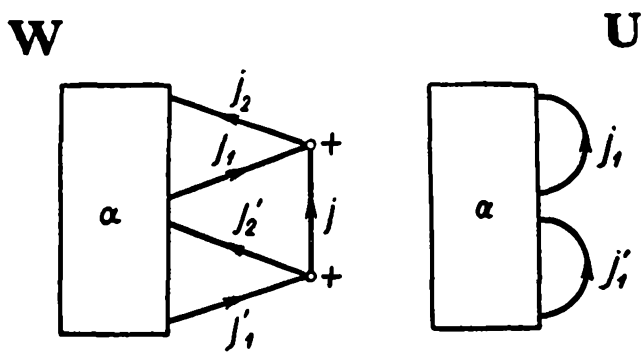

Figure 14.5

angular momenta is zero. Let us consider the jm-coefficient $W_\alpha(j_1 j_2;\ j_1' j_2';\ j)$ (diagram in Figure 14.5). If $j = 0$ the line j is dropped, as indicated in section 12. Two nodes, each with two lines, are then obtained. Dropping these nodes we find

$$W_\alpha(j_1 j_2;\ j_1' j_2';\ 0) =$$

$$= [(j_1)\ (j_1')]^{-\frac{1}{2}}\ \delta(j_1, j_2)\ \delta(j_1', j_2')\ U_\alpha(j_1 j_1'). \qquad (14.12)$$

One can see from diagram 14.5 how diagram U is obtained from the initial diagram W.

15. Summation of jm-coefficients

In calculations with jm-coefficients one is often required to sum the products of these coefficients over the parameters j. In this section we shall indicate how this summation may be carried out graphically with the help of diagrams. We shall state the problem as follows given the diagrams of the coefficients which are summed over, we wish to find the diagram of the coefficient which is the given sum

The diagram of the sum may be found from two rules which admit of simple graphical interpretation. The first is the "summation rule" of the jm-coefficient over the parameters j. Consider the diagram C in Figure 15.1. The corresponding

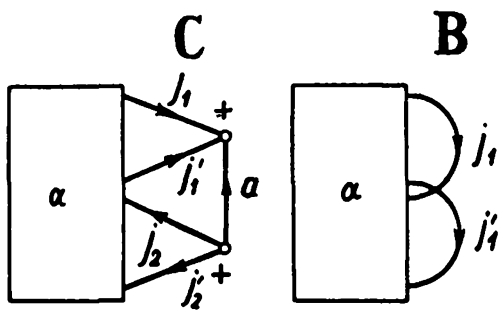

Figure 15.1

jm-coefficient may be written as follows

$$C_{\bar{a}}(j_1 j_1', j_2 j_2'; a) =$$

$$= \sum_{m_1 m_2 m_1' m_2' \mu} (-1)^{j_1 - m_1 + j_2 - m_2 + j_1' - m_1' + j_2' - m_2' + a - \mu} \times$$

$$\times A_\alpha \begin{pmatrix} j_1 & j_1'' & j_2 & j_2' \\ m_1 & m_1' & -m_2 & -m_2' \end{pmatrix} \begin{pmatrix} j_1 & j_1'' & a \\ -m_1 & -m_1' & -\mu \end{pmatrix} \begin{pmatrix} j_2 & j_2' & a \\ m_2 & m_2' & \mu \end{pmatrix}. \qquad (15.1)$$

For $j_2 = j_1$ and $j_2' = j_1'$ the jm-coefficient C may be summed over a with the aid of (5.6b)

$$\sum_a (a)\, C_a(j_1 j_1', j_1 j_1', a) = B_\alpha(j_1 j_1'), \qquad (15.2)$$

where

$$B_\alpha(j_1 j_1') = \sum_{m_1 m_1'} (-1)^{j_1 - m_1 + j' - m_1'} A_\alpha \begin{pmatrix} j_1 & j_1'' & j_1 & j_1'' \\ m_1 & m_1' & -m_1 & -m_1' \end{pmatrix}. \qquad (15.3)$$

The diagram of the jm-coefficient B is given in Figure 15.1. By comparing diagrams C and B one can formulate a convenient graphical rule for the summation of jm-coefficients over angular momentum quantum numbers. Let the nodes at the ends of the line representing the summation parameter j join corresponding lines and have the same orientation, and let one node join converging lines and the other diverging lines. To find the diagram of the sum one must cut the lines adjacent to

the line j, drop the part of the diagram containing j and contract corresponding lines in the remaining part.

The second rule is the "rule of multiplication" of a jm-coefficient and a j-coefficient. Let us consider the product of the jm-coefficient $C_\beta(l_1 l_2 l_3)$ and the j-coefficient $R_{\bar{\alpha}}(l_1 l_2 l_3)$ the diagrams of which (Figure 15.2) contain identically

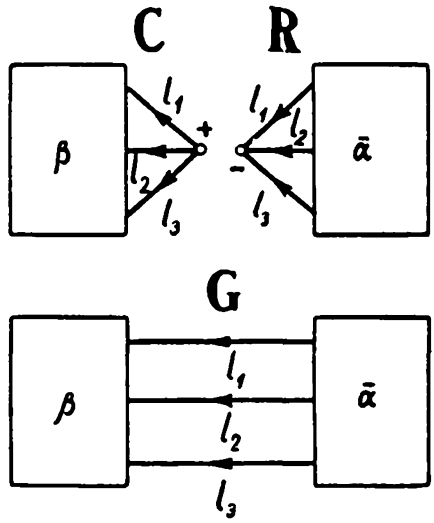

Figure 15.2

oriented nodes joining the lines l_1, l_2, l_3. From (14.5) the product of these coefficients may be represented by a single jm-coefficient $C_{\bar{\alpha}\beta}(l_1 l_2 l_3)$. The diagram of this jm-coefficient may be obtained by erasing identical nodes from diagrams C and R, followed by contraction of the two diagrams.

With these two rules summations may be carried out graphically in all cases. However, so as to avoid repeated application of the multiplication rule, it is more practical to derive a general rule for the summation of the products of coefficients over single angular momentum parameters.

Consider the sum of the products of the jm-coefficients C_{α_0} and the j-coefficients $W_{\bar{\alpha}_i}$ $(i = 1, 2, \ldots, k-1)$:

$$A = \sum_a (a)\, C_{\alpha_0}(j_1 j_1'; j_2 j_2'; a)\; W_{\bar{\alpha}_1}(j_2 j_2''; j_3 j_3'; a) \ldots$$
$$\ldots W_{\bar{\alpha}_{k-1}}(j_k j_k'; j_1 j_1'; a). \qquad (15.4)$$

The diagrams of C_{α_0} and $W_{\bar{\alpha}_i}$ are given in Figure 15.3. It is obvious that this representation of the diagrams is equivalent to the representation in Figure 15.1. The diagrams C_{α_0} and $W_{\bar{\alpha}_1}$ have identical and identically oriented nodes joining the lines j_2, j_2', a. In accordance with the "multiplication rule", the product $C_{\alpha_0} \cdot W_{\bar{\alpha}_1}$ may be expressed in the form of the jm-coefficient $C_{\alpha_0 \bar{\alpha}_1}$. The diagram of this jm-coefficient is given in Figure 15.4. One sees from this figure that the jm-coefficient $C_{\alpha_0 \bar{\alpha}_1}$ may be represented in the form $C_{\alpha_{01}}(j_1 j_1'; j_3 j_3''; a)$, where the block α_{01} is obtained by joining the lines j_2 and j_2'' in the blocks α_0 and $\bar{\alpha}_1$. Continuing this

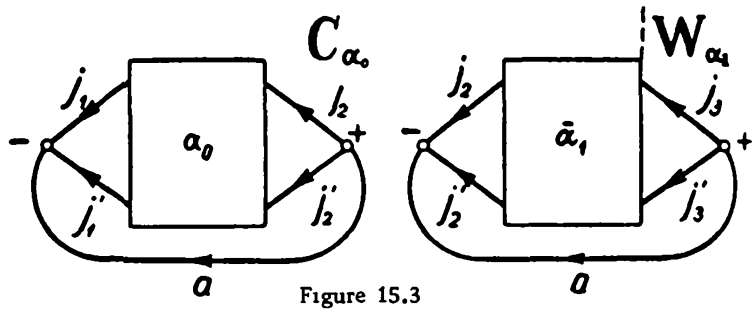

Figure 15.3

process, we can write the product under the summation sign in (15.4) as $C_{\alpha_{01}\cdot\ k-1}$ $(j_1 j_1'; j_1 j_1'; a)$. The summation of this jm-coefficient over a may be carried out by the "summation rule" (15.2). The diagram of the sum A will be of the form shown

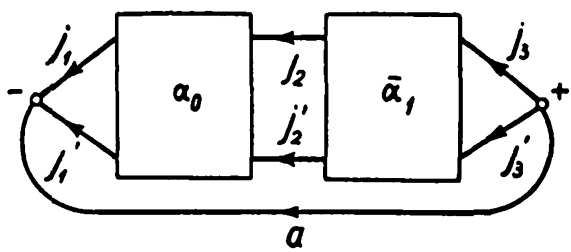

Figure 15.4

in Figure 15.5. The rule for obtaining this diagram may be formulated as follows. Cut the lines adjacent to the lines a in the individual diagrams, drop the parts containing the lines a and contract corresponding lines in the remaining parts. Further,

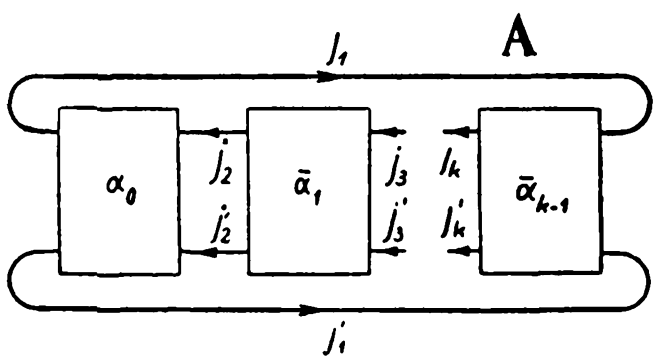

Figure 15.5

the lines and nodes adjacent to the lines a must be oriented correspondingly.

These rules of summation make it possible to sum any expression over angular momentum parameters. Concrete cases of such summations will be discussed later on (see Chapter VI).

Chapter IV

j-COEFFICIENTS AND THEIR PROPERTIES

In the mathematical apparatus of the theory of angular momentum an important part is played by products of Wigner coefficients which are summed over all magnetic quantum numbers. As noted in the preceding chapter, these quantities are called j-coefficients. In the sense of (11.6), they are invariant under rotation of the coordinate system. As we saw in section 13, they are used to express the coefficients of expansion of an arbitrary sum of products of Wigner coefficients in terms of generalized Wigner coefficients.

It is easily seen that the diagram of a j-coefficient, which does not have free lines, contains $2n$ nodes and $3n$ lines ($n = 1, 2, 3, \ldots$). The j-coefficient thus contains $3n$ parameters. The number of different j-coefficients with a given number of parameters $3n$ is determined by the number of different diagrams with $2n$ nodes. Further, only those diagrams which are separable on no less than four lines are significant, as from section 14 the j-coefficients represented by the other diagrams reduce to products of simpler j-coefficients. For given n all significant diagrams may be constructed either directly /Gutman and Budrite 1959/ or by recurrence from diagrams for $n - 1$ /Levinson and Chiplis 1958/.

For $n = 1$ we have the trivial j-coefficient, the triangular delta. For $n = 2$ there is one j-coefficient, it is the simplest nontrivial j-coefficient, and is described in section 16. The number of essentially different j-coefficients for $n = 3$, 4, 5 and 6 is 1, 2, 5 and 18 respectively.

By separating the diagram of an arbitrary j-coefficient, one can represent it in the form of a (generally) multiple sum of $6j$-coefficients. From the diagrams it is easily seen that single sums of this type (section 17) can only be of two kinds. Consequently, j-coefficients can be expressed in the form of single sums only for $n = 3$ and 4 (sections 18 and 19). When j-coefficients with more than twelve parameters are considered, multiple sums make their appearance (section 20 and appendices 3 and 4).

16 The $6j$-coefficient and its properties

The $6j$-coefficient, frequently called the Racah coefficient, is a product of four Wigner coefficients summed over all the magnetic quantum numbers (the form given below was obtained by Wigner /1937/, the W-coefficient examined by Racah /1942/ differs from the following by a phase factor (see appendix 1)).

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} = \sum_{m_i n_i} (-1)^{j_1 - m_1 + j_2 - m_2 + j_3 - m_3 + l_1 - n_1 + l_2 - n_2 + l_3 - n_3} \times$$

$$\times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & l_2 & l_3 \\ -m_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ n_1 & -n_2 & m_3 \end{pmatrix} \times$$

$$\times \begin{pmatrix} l_1 & j_2 & l_3 \\ -n_1 & -m_2 & -n_3 \end{pmatrix}. \qquad (16.1)$$

The graphical representation of this coefficient is easily obtained from the rules derived in section 12. It is given in Figure 16.1.

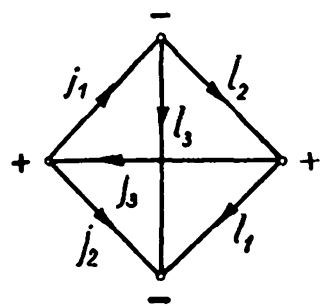

Figure 16.1.

Racah /1942/ reduced the expression (16.1) for the $6j$-coefficient to a single sum and obtained the following formula:

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} = (-1)^{j_1+j_2+l_1+l_2} \Delta(j_1 j_2 j_3) \Delta(j_1 l_2 l_3) \Delta(l_1 j_2 l_3) \Delta(l_1 l_2 j_3) \times$$

$$\times \sum_z (-1)^z \frac{(j_1+j_2+l_1+l_2+1-z)!}{z!(j_1+j_2-j_3-z)!(l_1+l_2-j_3-z)!(j_1+l_2-l_3-z)!} \times$$

$$\times \frac{1}{(l_1+j_2-l_3-z)!(j_2+l_3-j_1-l_1+z)!(j_3+l_3-j_2-l_2+z)!}. \qquad (16.2)$$

Here Δ is the triangle coefficient defined by (4.5) and z runs over all integral values which do not lead to negative arguments in the factorials. The sum (16.2) may be rewritten in the symmetrical and easily memorized form

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} = \Delta(j_1 j_2 j_3) \Delta(j_1 l_2 l_3) \Delta(l_1 j_2 l_3) \Delta(l_1 l_2 j_3) \times$$

$$\times \sum_z (-1)^z \frac{(z+1)!}{(z-j_1-j_2-j_3)!(z-j_1-l_2-l_3)!(z-l_1-j_2-l_3)!(z-l_1-l_2-j_3)!} \times$$

$$\times \frac{1}{(j_1+j_2+l_1+l_2-z)!(j_1+j_3+l_1+l_3-z)!(j_2+j_3+l_2+l_3-z)!}. \qquad (16.2a)$$

Like the Clebsch-Gordan coefficient, the $6j$-coefficient may be expressed in terms

of the hypergeometric function /Rose 1957/

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} = \Delta(j_1 j_2 j_3)\,\Delta(j_1 l_2 l_3)\,\Delta(l_1 j_2 l_3)\,\Delta(l_1 l_2 j_3)(-1)^{j_1+j_2+l_1+l_2}\times$$

$$\times \frac{(j_1+j_2+l_1+l_2+1)!}{(j_3+l_3-j_1-l_1)!(j_3+l_3-j_2-l_2)!(j_1+j_2-j_3)!(j_1+l_3-l_2)!(l_1+l_2-j_3)!(l_1+j_2-l_3)!}\times$$

$$\times {}_4F_3(-j_1-j_2+j_3, -l_1-l_2+j_3, -j_1-l_2+l_3, -l_1-j_2+l_3,$$

$$-j_1-j_2-l_1-l_2-1, j_3+l_3-j_1-l_1+1,$$

$$j_3+l_3-j_2-l_2+1; 1). \qquad (16.3)$$

The triad structure of the $6j$-coefficient is quite evident from the definition (16.1) and particularly from the diagram in Figure 16.1. The parameters form four triads which are the arguments of the four triangle coefficients in (16.2) and (16.3).

The symmetry properties of the $6j$-coefficient are as follows

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} = \begin{Bmatrix} j_a j_b j_c \\ l_a l_b l_c \end{Bmatrix} = \begin{Bmatrix} j_a l_b l_c \\ l_a j_b j_c \end{Bmatrix}, (a, b, c = 1, 2, 3) \qquad (16.4)$$

(16.4) gives the 24 symmetry properties found by Racah /1942/ and Wigner /1937/ These properties are easily found from expression (16.2) or with the aid of the diagram in Figure 16.1. To illustrate the symmetries, consider, for example,

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} = \begin{Bmatrix} j_2 j_1 j_3 \\ l_2 l_1 l_3 \end{Bmatrix}.$$

Evidently, this permutation changes nothing in (16.2), and in the diagram in Figure 16.1 it corresponds to a rotation of the figure about the line j_3 through 180°, with a subsequent change in signs of all the nodes and change in the directions of the lines l_1, l_2 and l_3. It is easily seen that these changes give the phase factor +1. To obtain certain other properties in (16.4) from (16.2) one must replace the summation parameter z by $z+$ (combination of the parameters). To obtain them from the graphical representation in Figure 16.1 it is necessary to re-draw the figure in such a way that j_3 and l_3 become the sides of a quadrilateral. This is done by redistributing the nodes on the closed line (the so-called Hamilton line). A more elegant method of obtaining the 24 symmetry properties (16.4) would be to give the diagram of the $6j$-coefficient the form of a tetrahedron, the symmetry group of which will then give these properties /Edmonds 1957/.

The sum (16.2) can be simplified when one of the parameters is zero, or when the sum of two parameters in some triad is equal to the third parameter of this triad (in both cases the triangle of the parameters stretches into a line). In the first case we have

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 0 \end{Bmatrix} =$$

$$= (-1)^{j_1+j_2+j_3} \{j_1 j_2 j_3\}\, \delta(j_1, l_2)\, \delta(j_2, l_1)\, [(j_1)(j_2)]^{-\frac{1}{2}}. \qquad (16.5)$$

In the second case

$$\begin{Bmatrix} j_1 j_2 & j_1+j_2 \\ l_1 l_2 & l_3 \end{Bmatrix} = (-1)^{j_1+j_2+l_1+l_2} \times$$

$$\times \left[\frac{(2j_1)!\,(2j_2)!\,(j_1+j_2+l_1+l_2+1)!\,(j_1+j_2+l_1-l_2)!\,(j_1+j_2-l_1+l_2)!}{(2j_1+2j_2+1)!\,(l_1+l_2-j_1-j_2)!\,(j_1+l_2-l_2)!\,(j_2-l_2+l_2)!} \times \right.$$

$$\left. \times \frac{(l_1+l_2-j_2)!\,(l_2+l_3-j_1)!}{(j_1+l_2+l_3+1)!\,(j_2+l_1-l_3)!\,(j_2-l_1+l_3)!\,(j_2+l_1+l_3+1)!} \right]^{\frac{1}{2}}. \qquad (16.6)$$

These formulas simplify the calculation of certain $6j$-coefficients.

The $6j$-coefficient satisfies certain recurrence relations /Biederharn 1953/ which are obtained from (24.9) by assigning particular values to certain parameters. We shall quote these relations. To increase a parameter by $\frac{1}{2}$ /Obi et al. 1953, Edmonds 1957/

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} = [(j_1+j_2+j_3+1)(-j_1+j_2+j_3) \times$$

$$\times (l_1+l_2+j_3+1)(l_1+j_3-l_2)]^{-\frac{1}{2}} \times$$

$$\times \left[-2j_3 \left[(l_1+j_2+l_3+1)(l_1+j_2-l_3) \right]^{\frac{1}{2}} \times \right.$$

$$\times \begin{Bmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ l_1-\frac{1}{2} & l_2 & l_3 \end{Bmatrix} + \left[(j_1+j_2-j_3+1)(j_1-j_2+j_3) \times \right.$$

$$\left. \left. \times (l_1+l_2-j_3+1)(j_3+l_2-l_1) \right]^{\frac{1}{2}} \begin{Bmatrix} j_1 j_2 j_3-1 \\ l_1 l_2 l_3 \end{Bmatrix} \right]; \qquad (16.7)$$

to increase a parameter by 1

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} = \left[(j_1-1)^2 (j_1+j_2+j_3+1)(j_1-j_2+j_3) \times \right.$$

$$\times (j_1+j_2-j_3)(-j_1+j_2+j_3+1)(j_1+l_2+l_3+1) \times$$

$$\left. \times (j_1-l_2+l_3)(j_1+l_2-l_3)(-j_1+l_2+l_3+1) \right]^{-\frac{1}{2}} \times$$

$$\times \left[-(2j_1-1) \left(2 \left[j_2(j_2+1) l_3(l_2+1) + j_3(j_3+1) l_3(l_3+1) - \right. \right. \right.$$

$$- j_1(j_1-1) l_1(l_1+1) \left. \right] - \left[j_2(j_2+1) + j_3(j_3+1) - j_1(j_1-1) \right] \times$$

$$\left. \times \left[l_2(l_2+1) + l_3(l_3+1) - j_1(j_1-1) \right] \right) \cdot \begin{Bmatrix} j_1-1 & j_2 j_3 \\ l_1 & l_2 l_3 \end{Bmatrix} -$$

$$- j_1 \left[(j_1+j_2+j_3)(j_1-j_2+j_3-1)(j_1+j_2-j_3-1) \times \right.$$

$$\times(-j_1+j_2+j_3+2)(j_1+l_2+l_3)(j_1+l_2-l_3-1)\times$$

$$\times(j_1-l_2+l_3-1)(-j_1+l_2+l_3+2)\Big]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} j_1 & -2 & j_2 & j_3 \\ l_1 & & l_2 & l_3 \end{Bmatrix}\Bigg]. \qquad (16.8)$$

The $6j$-coefficient also possessses the following property /Regge 1959/

$$\begin{Bmatrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{Bmatrix} =$$

$$= \begin{Bmatrix} j_1 & \frac{1}{2}(j_2+l_2+j_3-l_3) & \frac{1}{2}(j_3+l_3+j_2-l_2) \\ l_1 & \frac{1}{2}(j_2+l_2+l_3-j_3) & \frac{1}{2}(j_3+l_3+l_2-j_2) \end{Bmatrix}. \qquad (16.9)$$

This relation enables us to replace one value of a parameter by another, whereas from (16.4) one can only change their position. (16.9) is therefore more of a functional relation than symmetry as is usually understood, as by symmetry is meant permutation of the arguments. For the case in which two columns in the $6j$-coefficient are such that the sum of three parameters equals the fourth, this coefficient may be brought to the form (16.5) by using (16.9). Thus

$$\begin{Bmatrix} \frac{5}{2} & 3 & \frac{1}{2} \\ 2 & \frac{3}{2} & 1 \end{Bmatrix} = \begin{Bmatrix} \frac{5}{2} & 0 & \frac{5}{2} \\ 2 & \frac{3}{2} & 2 \end{Bmatrix}.$$

Let us list the available tables of $6j$-coefficients. The tables of Simon et al. /1954/ (see appendix 1) give approximate values of $W(j_1 j_2 l_2 l_1, j_3 l_3)$ for

$$j_2,\ l_1 = 0\left(\frac{1}{2}\right)\frac{9}{2}$$

$$j_3 = 0\left(\frac{1}{2}\right)3$$

$$j_1,\ l_2 = 0\left(\frac{1}{2}\right)\frac{15}{2}$$

$$l_3 = 0(1)8$$

to nine places of decimals calculated with the aid of electronic computers. The tables of Obi et al. give exact values of W in the form of fractions for all integral values of the parameters within the limits

$$j_1,\ j_2,\ l_2 \geq l_1,\ j_3 \geq l_3$$

$$l_1+l_2+l_3 \leq 15;\quad l_3,\ j_3 = 0(1)7$$

/Obi et al. 1953/ and for integral j_3, l_1, l_3 and half-integral j_1, l_2, j_3 within the limits

$$j_3 \geq l_1,\ j_1 \geq l_2 \quad \text{for} \quad j_3 = l_1;$$

$$l_1 + l_2 + l_3 \leq 15;$$

$$l_1 = 0(1)7;\ l_2,\ j_3 = \frac{1}{2}\left(\frac{1}{2}\right)\frac{13}{2};$$

/Obi et al. 1954, 1955/.

Boys and Sahni /1954/ give tables of numerical values of the quantity U which is closely related to the $6j$-coefficient (see appendix 1) for the parameters

$$j_1, j_2, l_1, l_2 = 0\left(\frac{1}{2}\right)\frac{5}{2}$$

$$j_3, l_3 = 0\left(\frac{1}{2}\right)5.$$

Biedenharn /1952/ gives tables of the function W for

$$j_1 = 0(1)4;\ l_2 = 0(1)5;\ j_3, l_1 = 0\left(\frac{1}{2}\right)\frac{9}{2};$$

$$j_3 = \frac{1}{2}\left(\frac{1}{2}\right)3;\ l_3 = 0(1)8.$$

In some cases it is possible to use algebraic formulas for the $6j$-coefficient for certain values of one of the parameters. These formulas are easily obtained from the expression (16.2), which was transformed by Sato /1955/ to the following form

$$\left\{ \begin{matrix} j_1 + k_j & j_1 & j_3 \\ l_1 & l_1 + k_l & l_3 \end{matrix} \right\} = (-1)^{D+k_j+k_l} \times$$

$$\times \left[\frac{(j_3 + k_j)^{(2k_j)} A^{(k_j - k_l)} (B + k_j - k_l)^{(k_j - k_l)} (C + k_j + k_l)^{(k_j + k_l)}}{(j_3 + k_l)!\,(j_3 - k_l)!\,(2j_1 + j_3 + k_j + 1)^{(2j_3+1)}} \times \right.$$

$$\left. \times \frac{(D + k_j + k_l + 1)^{(k_j + k_l)}}{(2l_1 + j_3 + k_l + 1)^{(2j_3+1)}} \right]^{\frac{1}{2}} \times$$

$$\times \left[(j_3 + k_l)(A - k_j + k_l) B - \right.$$

$$\left. - (j_3 - k_l) C (D + k_j + k_l + 1)^{(-1)} \right]^{(j_3 - k_l)}. \qquad (16.10)$$

Here

$$V^{(k)} = V(V-1)\ldots(V-k+1)$$
$$V^{(-k)} = (V+1)\cdot\cdot\cdot(V+k),$$
$$A = -j_1 + l_1 + l_3,$$
$$B = j_1 - l_1 + l_3,$$
$$C = j_1 + l_1 - l_3,$$
$$D = j_1 + l_1 + l_3. \qquad (16.11)$$

Formula (16.10) has been written for the case $k_j \geqslant |k_i|$. If this condition is not satisfied, we obtain the required expressions for $k_i \geqslant |k_j|$ by changing j_1 to l_1 and k_j to k_i. The last factor is expanded symbolically by the binomial theorem.

Algebraic formulas obtained from (16.10) for $j_3 = 3$, $\frac{7}{2}$, 4 and $\frac{9}{2}$ are given in Obi et al. /1954, 1955/ and Sato /1955/. They are given in a slightly different form for $j_3 = \frac{1}{2}$, 1, $\frac{3}{2}$ and 2 by Biedenharn et al. /1952/, and for $j_3 = \frac{5}{2}$ by Edmonds and Flowers /1952/. Asymptotic formulas for the $6j$-coefficient for large parameters may be found in Biedenharn /1953/ and in Brussard and Tolhoek /1957/.

17. 3 nj-coefficients of the first and second kinds

The $6j$-coefficient is the simplest nontrivial j-coefficient. In a certain sense it is possible to regard it as the principal j-coefficient, and to express all j-coefficients as sums of products $6j$-coefficients. In this section we will consider certain single sums of products of $6j$-coefficients, which represent j-coefficients of the two types called 3 nj-coefficients of the first and second kind /Levinson and Vanagas 1957/.

The sum of a product of $6j$-coefficients which represents a 3 nj-coefficient of the first kind is of the form

$$\begin{Bmatrix} j_1 & j_2 \cdots j_n \\ l_1 & l_2 \ldots l_n \\ k_1 & k_2 \ldots k_n \end{Bmatrix} = \sum_x (x)(-1)^{R_n+(n-1)x} \times$$

$$\times \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-1} & k_{n-1} & x \\ k_n & j_n & l_{n-1} \end{Bmatrix} \begin{Bmatrix} j_n & k_n & x \\ j_1 & k_1 & l_n \end{Bmatrix} \tag{17.1}$$

and the sum which gives a 3 nj-coefficient of the second kind is

$$\begin{bmatrix} j_1 & j_2 \cdots j_n \\ l_1 & l_2 \ldots l_n \\ k_1 & k_2 \ldots k_n \end{bmatrix} = \sum_x (x)(-1)^{R_n+nx} \times$$

$$\times \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-1} & k_{n-1} & x \\ k_n & j_n & l_{n-1} \end{Bmatrix} \begin{Bmatrix} j_n & k_n & x \\ k_1 & j_1 & l_n \end{Bmatrix}. \tag{17.2}$$

Here

$$R_n = \sum_{i=1}^{n} (j_i + k_i + l_i). \tag{17.3}$$

In 3 nj-coefficients of the first kind, triads are formed by the following sets of parameters $(j_1 l_1 j_2)$, $(j_2 l_2 j_3)$, $\ldots$, $(j_{n-1} l_{n-1} j_n)$, $(j_n l_n k_1)$, $(k_1 l_1 k_2)$, $\ldots$, $(k_{n-1} l_{n-1} k_n)$, $(k_n l_n j_1)$, in 3 nj-coefficients of the second kind we have $(j_1 l_1 j_2)$, $(j_2 l_2 j_3)$, $\ldots$, $(j_{n-1} l_{n-1} j_n)$, $(j_n l_n j_1)$, $(k_1 l_1 k_2)$, $(k_2 l_2 k_3)$, $\ldots$, $(k_{n-1} l_{n-1} k_n)(k_n l_n k_1)$. It is useful

to note that the expansions (17.1) and (17.2) differ only in the phase factor $(-1)^x$ and in the distribution of the parameters j_1 and k_1 in the last factor under the summation sign. This leads to different intersection properties of the lines j_1 and k_1 in the diagrams representing $3nj$-coefficients of the first and second kind (Figures 17.1 and 17.2 respectively).

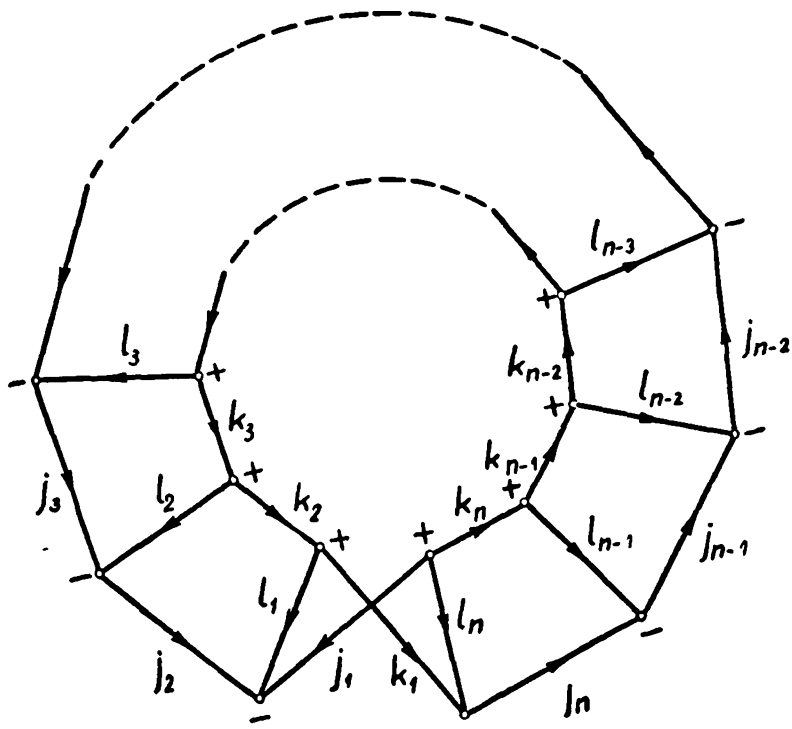

Figure 17.1

Obviously, there need not be only one pair of intersecting lines. However, by deforming the diagram one can erase two pairs of intersecting lines without essentially changing the given sum. An uneven number of intersections therefore leads to Figure 17.1, and an even number to Figure 17.2. Hence it follows that, apart from those we have examined, no other forms of single sums exist.

The symmetry properties of $6j$-coefficients (16.4) give the following symmetry properties for $3nj$-coefficients of the first and second kinds respectively

$$\left\{\begin{matrix} j_1 & j_2 \cdots j_n \\ l_1 & l_2 \ldots l_n \\ k_1 & k_2 \ldots k_n \end{matrix}\right\} = \left\{\begin{matrix} k_1 & k_2 & \ldots k_n \\ l_1 & l_2 \ldots l_n \\ j_1 & j_2 \cdots j_n \end{matrix}\right\} =$$

$$= \left\{\begin{matrix} j_2 \cdots j_n & k_1 \\ l_2 \ldots l_n & l_1 \\ k_2 \ldots k_n & j_1 \end{matrix}\right\} = \left\{\begin{matrix} k_1 & j_n \cdots j_2 \\ l_n & l_{n-1} \ldots l_1 \\ j_1 & k_n \ldots k_2 \end{matrix}\right\}, \qquad (17.4)$$

$$\begin{bmatrix} j_1 & j_2 \cdots & j_n \\ l_1 & l_2 \cdots & l_n \\ k_1 & k_2 \cdots & k_n \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \cdots & k_n \\ l_1 & l_2 \cdots & l_n \\ j_1 & j_2 \cdots & j_n \end{bmatrix} =$$

$$= \begin{bmatrix} j_2 \cdots j_n & j_1 \\ l_2 \cdots l_n & l_1 \\ k_2 \cdots k_n & k_1 \end{bmatrix} = \begin{bmatrix} j_1 & j_n \cdots j_2 \\ l_n & l_{n-1} \cdots l_1 \\ k_1 & k_n \cdots k_2 \end{bmatrix}. \qquad (17.5)$$

To obtain the first set of relations one must permute the rows in the $6j$-coefficients in (17.1) and (17.2) in the columns which do not contain the summation parameter.

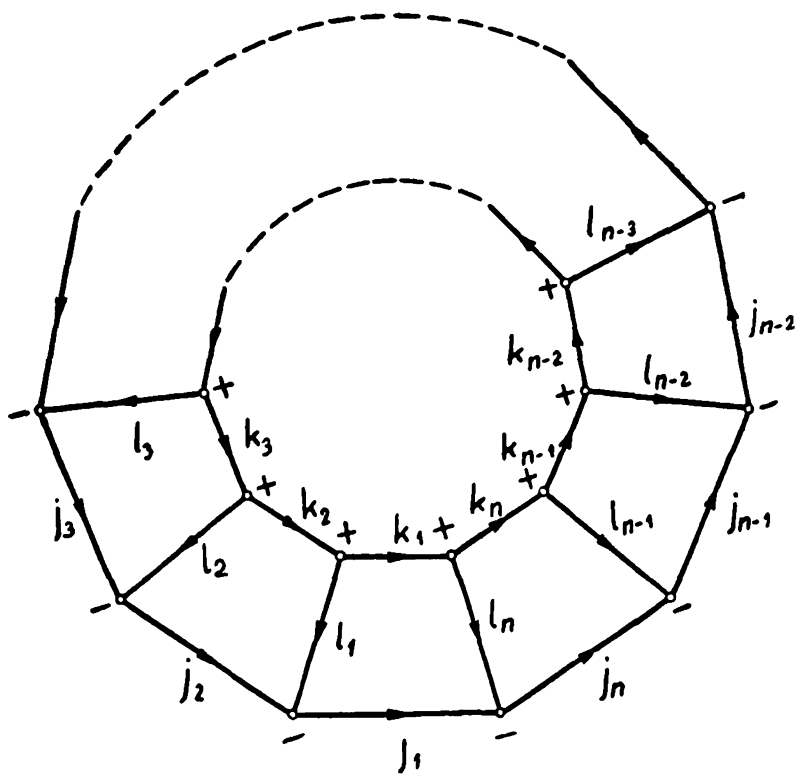

Figure 17.2

To obtain the second set of relations the first $6j$-coefficient must be transferred to the last position. Finally, to obtain the last of these relations one must write all the $6j$-coefficients in reverse order and permute the rows and columns, without changing the columns containing the summation parameter.

It follows immediately from the triad structure that the sum of four parameters taken from two columns in the outer rows is an integer.

If one of the parameters in the $3nj$-coefficient vanishes then we obtain, taking

(16.5) into account,

$$\begin{Bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} \quad 0 \\ k_1 & \cdots & k_{n-1} & k_n \end{Bmatrix} = \left[(j_1)(k_1)\right]^{-\frac{1}{2}} \delta(j_n, k_1)\,\delta(k_n, j_1) \times$$

$$\times \begin{Bmatrix} j_1 & \cdots & j_{n-1} \\ & l_1 & \cdots \quad l_{n-1} \\ k_1 & \cdots & k_{n-1} \end{Bmatrix}, \tag{17.6}$$

$$\begin{Bmatrix} j_1 & \cdots & j_{n-1} & 0 \\ & l_1 & \cdots & l_{n-1} \quad l_n \\ k_1 & \cdots & k_{n-1} & k_n \end{Bmatrix} = \left[(l_n)(l_{n-1})\right]^{-\frac{1}{2}} \times$$

$$\times \delta(l_n, k_1)\,\delta(l_{n-1}, j_{n-1})(-1)^{R_{n-1}+nk_n-j_1+k_{n-1}+j_{n-1}} \times$$

$$\times \begin{Bmatrix} j_1 & k_1 & k_n \\ k_2 & j_2 & l_1 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-2} & k_{n-2} & k_n \\ k_{n-1} & j_{n-1} & l_{n-2} \end{Bmatrix}, \tag{17.7}$$

$$\begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} \quad 0 \\ k_1 & \cdots & k_{n-1} & k_n \end{bmatrix} = \left[(j_1)(k_1)\right]^{\frac{1}{2}} \delta(j_n, j_1)\,\delta(k_n, k_1) \times$$

$$\times \begin{bmatrix} j_1 & \cdots & j_{n-1} \\ & l_1 & \cdots \quad l_{n-1} \\ k_1 & \cdots & k_{n-1} \end{bmatrix}, \tag{17.8}$$

$$\begin{bmatrix} j_1 & \cdots & j_{n-1} & 0 \\ & l_1 & \cdots & l_{n-1} \quad l_n \\ k_1 & \cdots & k_{n-1} & k_n \end{bmatrix} = \left[(l_n)(l_{n-1})\right]^{-\frac{1}{2}} \times$$

$$\times \delta(j_{n-1}, l_{n-1})\,\delta(l_n, j_1)(-1)^{R_{n-1}+(n+1)k_n-k_1+k_{n-1}+l_{n-1}} \times$$

$$\times \begin{Bmatrix} j_1 & k_1 & k_n \\ k_2 & j_2 & l_1 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-2} & k_{n-2} & k_n \\ k_{n-1} & j_{n-1} & l_{n-2} \end{Bmatrix}. \tag{17.9}$$

From the symmetry properties (17.4) and (17.5), the first row can be interchanged with the third, consequently, (17.7) and (17.9) also cover the case where the vanishing parameter is in the third row. Vanishing of the parameter in the middle row, as we see, leads to a $3(n-1)j$-coefficient, i.e., reduces the number of parameters by three.

For $n = 1$, from (17.1) and (17.2) we have the trivial expressions

$$\begin{Bmatrix} j_1 \\ & l_1 \\ k_1 \end{Bmatrix} = (-1)^{j_1+k_1-l_1}\{j_1 l_1 k_1\}, \tag{17 10a}$$

$$\begin{bmatrix} j_1 & & \\ & l_1 & \\ & & k_1 \end{bmatrix} = \delta(l_1, 0)\left[(j_1)(k_1)\right]^{\frac{1}{2}}. \tag{17.10b}$$

For $n = 2$

$$\begin{Bmatrix} j_1 & j_2 & \\ & l_1 & l_2 \\ k_1 & k_2 & \end{Bmatrix} = (-1)^{j_1+j_2+k_1+k_2} \begin{Bmatrix} j_1 & j_2 & l_1 \\ k_1 & k_2 & l_2 \end{Bmatrix}, \tag{17.11a}$$

$$\begin{bmatrix} j_1 & j_2 & \\ & l_1 & l_2 \\ k_1 & k_2 & \end{bmatrix} = \delta(l_1, l_2)(l_1)^{-1} \times$$

$$\times (-1)^{j_1-j_2-k_1+k_2} \{j_1 j_2 l_1\} \{k_1 k_2 l_1\}. \tag{17.11b}$$

A comparison of (17.11a) with (A. 1.2a) [A = appendix] shows that the $6j$-coefficient is obtained as a particular case from the $3nj$-coefficient with the same phase as Racah's W. Equation (17.11b) shows that the $6j$-coefficient of the second kind is trivial.

18. The $9j$-coefficient and its properties

If in (17.1) $n = 3$, then, introducing a more convenient notation (following Wigner /1937/)

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ & l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 & \end{Bmatrix} = \begin{Bmatrix} l_1 & j_1 & j_2 \\ k_1 & l_3 & j_3 \\ k_2 & k_3 & l_2 \end{Bmatrix}, \tag{18.1}$$

we obtain

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{Bmatrix} = \sum_x (x)(-1)^{2x} \times$$

$$\times \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & k_3 & x \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ j_2 & x & k_2 \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & k_3 \\ x & j_1 & l_1 \end{Bmatrix}. \tag{18.2}$$

Thus the 3 nj-coefficient of the first kind for $n = 3$ is the well-known $9j$-coefficient, which is sometimes referred to as Fano's X-coefficient or Wigner's $9j$-coefficient.

If in (17.2) $n = 3$, then we have

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ & l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 & \end{bmatrix} = \sum_x (x)(-1)^{R_3-x} \begin{Bmatrix} j_1 & l_1 & j_2 \\ k_2 & x & k_1 \end{Bmatrix} \times \begin{Bmatrix} j_2 & l_2 & j_3 \\ k_3 & x & k_2 \end{Bmatrix} \begin{Bmatrix} j_3 & l_3 & j_1 \\ k_1 & x & k_3 \end{Bmatrix} \tag{18.3}$$

The j-coefficients (18.2) and (18.3) are represented in Figures 18.1 and 18.2 respectively. It is therefore evident that the $9j$-coefficient of the second kind (18.3) is not

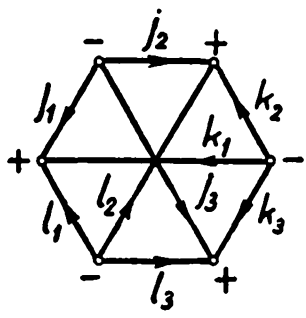

Figure 18.1

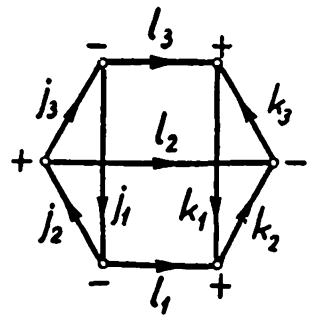

Figure 18.2

independent, since, as explained in section 14, diagram 18.2 is separable on the three lines l_1, l_2 and l_3. This gives

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{bmatrix} = (-1)^{2l_1+2k_1} \begin{Bmatrix} l_1 & l_2 & l_3 \\ k_3 & k_1 & k_2 \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ j_3 & j_1 & j_2 \end{Bmatrix}. \tag{18.4}$$

The $9j$-coefficient of the first kind (18.2), which is simply referred to as the $9j$-coefficient, possesses a high degree of symmetry. Its symmetry properties may be written as follows /Wigner 1937, Jahn and Hope 1954/

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{Bmatrix} = \begin{Bmatrix} j_1 & l_1 & k_1 \\ j_2 & l_2 & k_2 \\ j_3 & l_3 & k_3 \end{Bmatrix} = (-1)^{R_3} \begin{Bmatrix} j_2 & j_1 & j_3 \\ l_2 & l_1 & l_3 \\ k_2 & k_1 & k_3 \end{Bmatrix} = (-1)^{R_3} \begin{Bmatrix} l_1 & l_2 & l_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{Bmatrix},$$

$$R_3 = \sum_{i=1}^{3} (j_i + l_i + k_i). \tag{18.5}$$

They may be found, for instance, with the aid of diagram 18.1. This gives 72 symmetry properties in all. They show that the $9j$-coefficient does not change upon transposition of the parameters relative to the two diagonals and upon an even permutation of the columns or rows, but that upon an odd permutation of the columns or rows it is multiplied by the phase factor $(-1)^{R_3}$, where R_3 is the sum of all nine parameters. From the latter it follows that, when two rows or columns are equal, the following relation holds true

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{Bmatrix} = 0, \quad \text{if } k_1 + k_2 + k_3 = \text{is odd} \tag{18.6}$$

It is readily seen that in the diagram of the $9j$-coefficient it is possible to change the directions of all lines and the signs of all nodes without changing the phase factor. If one of the parameters of the $9j$-coefficient vanishes, then from (18.2) and (16.5) it

follows that

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & 0 \end{Bmatrix} = \delta(j_3, l_3)\,\delta(k_1, k_2)\left[(j_3)(k_1)\right]^{-\frac{1}{2}} \times$$

$$(-1)^{l_1+j_2+k_1+j_3} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & k_1 \end{Bmatrix}. \qquad (18.7)$$

Since the $9j$-coefficients are frequently encountered in calculations, we shall list the available tables of these coefficients. The tables of Smith and Stevenson /1957/ give values of $9j$-coefficients to 10 places of decimals for values of the parameters

$$j_1 = \tfrac{1}{2}(1)\tfrac{3}{2}; \quad j_2, k_1 = \tfrac{1}{2}(1)\tfrac{5}{2}; \quad k_2 = \tfrac{1}{2}(1)\tfrac{7}{2};$$

$$j_3, k_3, l_1, l_2 = 0(1)7.$$

The supplement to these tables /Smith 1958/ gives

$$j_1 = \tfrac{3}{2}\left(\tfrac{1}{2}\right)\tfrac{7}{2}; \quad j_2 = \tfrac{5}{2}\left(\tfrac{1}{2}\right)\tfrac{7}{2}; \quad j_3 = \tfrac{7}{2}; \quad l_1 = \tfrac{3}{2}\left(\tfrac{1}{2}\right)\tfrac{7}{2}.$$

Tables of $9j$-coefficients in fractional form have been computed by Sharp and his collaborators /Kennedy et al. 1954/, covering values of the parameters

1) $j_2 = k_2, j_3 = k_3 = \tfrac{1}{2}\left(\tfrac{1}{2}\right)5; \quad j_1 \leqslant k_1 = 1.2.$

2) $j_2 = k_2 = \tfrac{1}{2}(1)\tfrac{5}{2}; \quad j_3 = k_3 = \tfrac{3}{2}(1)\tfrac{5}{2}; \quad j_1 = k_1 = 3.4.$

3) $j_1 = k_1 = 1(1)4; \quad j_2 = \tfrac{1}{2}; \quad k_2 = \tfrac{3}{2}, \tfrac{5}{2};$

$j_3 = k_3 = \tfrac{1}{2}(1)\tfrac{9}{2}.$

Sharp et al. /1954/ give these coefficients for values of the parameters

1) $j_2 = k_2, j_3 = k_3 = 1\left(\tfrac{1}{2}\right)5; \; j_1 = k_1 = 1,2.$

$l_1, l_2, l_3 = 2(2)8$

2) $j_1 = 1, \; k_1 = 2, \; j_2 = k_2 \leqslant j_3 = k_3 = 1\left(\tfrac{1}{2}\right)5.$

Values of the quantity A (see appendix 1), which differs from the $9j$-coefficient by a simple factor, may be found in Kennedy and Cliff /1955/ in the form of decimals and fractions for

$$j_1, \; k_1 = 0(1)5, \; j_2 = k_2 = \tfrac{1}{2}.$$

It may be found useful to refer to the tables of Matsunobu and Takebe /1955/, which contain algebraic formulas and numerical values in the form of fractions for the $9j$-coefficient for $j_2 = k_2 = \frac{1}{2}$, $j_1, k_1 = 0(1)4$.

If in (17.1) and (17.2) $n=4$, we obtain

$$\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} = \sum_x (x)(-1)^{R_4-x} \times$$

$$\times \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{Bmatrix} \begin{Bmatrix} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{Bmatrix} \begin{Bmatrix} j_4 & k_4 & x \\ j_1 & k_1 & l_4 \end{Bmatrix}, \tag{19.1}$$

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \sum_x (x)(-1)^{R_4} \times$$

$$\times \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{Bmatrix} \begin{Bmatrix} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{Bmatrix} \begin{Bmatrix} j_4 & k_4 & x \\ k_1 & j_1 & l_4 \end{Bmatrix}. \tag{19.2}$$

(19.1), which is a $12j$-coefficient of the first kind, was introduced by Jahn and Hope /1954/ and studied by Ord-Smith /1954/. It is usually referred to simply as the $12j$-coefficient.

It is convenient to use the following more symmetric quantity instead of (19.2)

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = (-1)^{j_1-l_2-l_3+l_4} \sum_x (x) \times$$

$$\times \begin{Bmatrix} k_1 & k_3 & x \\ j_3 & j_1 & l_1 \end{Bmatrix} \begin{Bmatrix} k_3 & k_4 & x \\ j_3 & j_1 & l_2 \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & x \\ j_4 & j_2 & l_3 \end{Bmatrix} \begin{Bmatrix} k_3 & k_4 & x \\ j_4 & j_2 & l_4 \end{Bmatrix}, \tag{19.3}$$

which is related to (19.2) in the following way

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = (-1)^{-j_1+j_2+j_3-j_4+k_1-k_2-k_3+k_4} \times$$

$$\times \begin{bmatrix} k_3 & j_1 & k_1 & j_2 \\ l_2 & l_1 & l_3 & l_4 \\ k_4 & j_3 & k_2 & j_4 \end{bmatrix}. \tag{19.4}$$

This quantity was introduced by Elliott and Flowers /1955/ and studied by Sharp /1955/ in a somewhat different notation, related to the present one /Vanagas and Chiplis 1958/ by

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \begin{pmatrix} \cdot & k_4 j_3 & l_2 \\ k_1 & \cdot & l_1 j_1 \\ j_2 & l_4 & \cdot & k_3 \\ l_3 & j_4 & k_2 & \cdot \end{pmatrix}. \tag{19.5}$$

We shall call this quantity the $12j$-coefficient of the second kind. It is convenient to write it as follows, in order to examine the triad structure

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix}, \quad \begin{matrix} j_2 & j_3 \\ l_2 & l_3 \end{matrix} \tag{19.6}$$

here the triads can readily be seen.

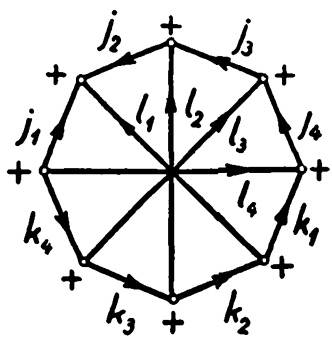

Figure 19.1a

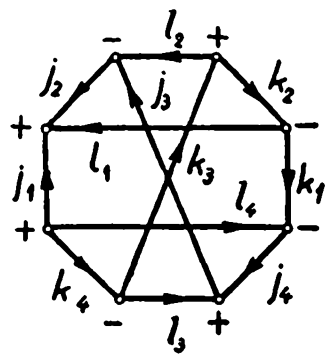

Figure 19.1b

The diagrams which represent $12j$-coefficients on a plane are given in Figures 19.1a, 19.1b and 19.2. We see from these that neither the $12j$-coefficient of the first kind nor the $12j$-coefficient of the second kind can be simplified, as their diagrams cannot be separated on fewer than four lines.

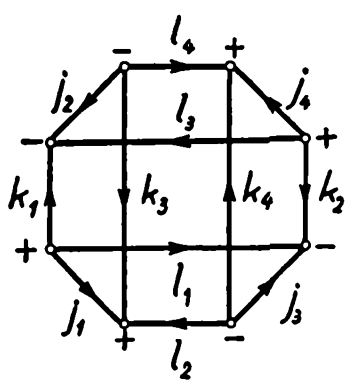

Figure 19.2

It should be noted that a $12j$-coefficient of the first kind may be drawn on a plane as a polygon by the two methods indicated in Figures 19.1a and 19.1b. as a consequence of the symmetry properties which follows directly from (17.4). Hamilton's line, which should form the perimeter of a polygon when passing from diagram 19.1b to 19.1a, consists of the lines j_1, j_2, j_3, j_4, k_1, k_2, k_3, k_4.

Like the $9j$-coefficient, the $12j$-coefficient of of the second kind possesses symmetry properties in addition to those of the general $3nj$-coefficient. These may easily be examined by representing the diagram of the $12j$-coefficient of the second kind in the form of a cube /Vanagas and Chiplis 1958/. The following symmetries are thus obtained

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \begin{bmatrix} j_4 & j_3 & j_2 & j_1 \\ l_4 & l_3 & l_2 & l_1 \\ k_4 & k_3 & k_2 & k_1 \end{bmatrix} \tag{19.7a}$$

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} = \begin{bmatrix} l_4 & l_2 & l_3 & l_1 \\ j_4 & j_2 & j_3 & j_1 \\ k_4 & k_2 & k_3 & k_1 \end{bmatrix} = \tag{19.7b}$$

$$= \begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_2 & l_1 & l_4 & l_3 \\ k_3 & k_4 & k_1 & k_2 \end{bmatrix}. \tag{19.7c}$$

This means that without changing the phase factor it is possible to a) interchange the first and fourth columns and the second and third columns simultaneously, b) carry out any permutation of the rows (for an odd permutation one must permute the first and fourth or the second and third columns simultaneously), c) permute the parameters l_1, l_2 and l_3, l_4, permuting the groups of parameters k_1, k_2 and k_3, k_4 simultaneously. (19.7) indicates that the $12j$-coefficient of the second kind possesses 24 symmetry properties whereas from (17.4) it follows that the $12j$-coefficient of the first kind possesses only 16 symmetry properties. Remembering that, as pointed out in section 17, the sum of any four parameters in two columns is an integer, we see from (19.4) that in the notation of (19.3) for the $12j$-coefficient of the second kind the sum of the parameters in a row is an integer.

If one of the parameters of the $12j$-coefficient vanishes, we obtain

$$\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ & l_1 & l_2 & l_3 & 0 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} = \delta(j_4, k_1)\,\delta(k_4, j_1)\left[(j_1)(k_1)\right]^{-\frac{1}{2}} \times$$

$$\times \begin{Bmatrix} j_1 & l_1 & j_2 \\ k_3 & k_2 & l_2 \\ l_3 & k_1 & j_3 \end{Bmatrix}, \tag{19.8}$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 & 0 \\ & l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} = \delta(l_4, k_1)\,\delta(l_3, j_3)\left[(l_3)(l_4)\right]^{-\frac{1}{2}} \times$$

$$\times (-1)^{k_1-k_2+l_2+l_3-j_3+l_1} \begin{Bmatrix} j_1 & j_2 & l_1 \\ k_2 & k_1 & k_4 \end{Bmatrix} \begin{Bmatrix} j_2 & j_3 & l_2 \\ k_3 & k_2 & k_4 \end{Bmatrix}, \tag{19.9}$$

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & 0 \end{bmatrix} = \delta(j_4, l_4)\,\delta(j_3, l_2)\left[(l_2)(l_4)\right]^{-\frac{1}{2}} \times$$

$$\times (-1)^{j_1+l_1-j_1-l_1} \begin{Bmatrix} l_1 & l_2 & k_2 \\ k_3 & k_1 & j_1 \end{Bmatrix} \begin{Bmatrix} k_2 & l_3 & l_4 \\ j_2 & k_3 & k_1 \end{Bmatrix}. \tag{19.10}$$

The above symmetry properties of the $12j$-coefficient of the second kind show that

in the latter as well as in the $6j$- and $9j$-coefficients all parameters are equivalent, i.e., any parameter may be placed in any position. We have therefore cited only one formula for the $12j$-coefficient of the second kind.

$3nj$-coefficients of the first and second kind exhaust all possible independent quantities with 6, 9 and 12 parameters. This may be seen by drawing all possible diagrams with four, six and eight nodes. This does not hold true for cases where the number of parameters is greater than 12. One is therefore obliged to turn to methods which enable one to find all possible independent quantities with the given number of parameters. We shall begin our examination of $15j$-coefficients with a brief exposition of these methods.

20. Methods of studying j-coefficients, $15j$-coefficients

To determine the number and type of $3nj$-coefficients for a given value of n, one must construct all possible diagrams with $3n$ lines separable on no fewer than four lines, and class them in sets reducible to a single form. The number of such sets will give the number of $3nj$-coefficients with the given number of parameters. In this study the principal aim is to construct all possible diagrams of this kind. This may be carried out by two methods, which were suggested by Levinson and Chiplis /1958/ and Gutman and Budrite /1959/ respectively.

The first of these is the recursion method. The diagrams of the $3nj$-coefficients are obtained from the diagrams of the $3(n-1)j$-coefficients in the following way two additional nodes are placed on any two lines of the diagram of the $3(n-1)j$-coefficient and joined by a line. To ensure that the diagram obtained for the $3nj$-coefficient will not be separable on fewer than four lines, one should initially take a diagram of a $3(n-1)j$-coefficient which is separable on no fewer than three lines. Joining in this way all possible pairs of lines in the diagrams of all $3(n-1)j$-coefficients, we obtain the diagrams of all $3nj$-coefficients. After selecting those which are separable on no fewer than four lines, and identifying similar diagrams, we obtain all essentially new diagrams of $3nj$-coefficients. Utilizing the symmetry of the diagrams of $3(n-1)j$-coefficients, it is possible to reduce the number of pairs of joined lines in an obvious way. For example, for $n=1$ we merely have the one diagram of a triangular delta (Figure 12.17). As all the lines in this diagram are equivalent, it is sufficient to join any pair of lines. Joining the mid-points of, say, the lines j_1 and j_3, we obtain the diagram of a $6j$-coefficient (Figure 16.1).

We will illustrate the second method by the construction of diagrams with six nodes. The construction always begins with the group of four nodes A, B_1, B_2, B_3 (Figure 20.1), as this part will occur in all diagrams separable on no fewer than four lines. In our example we shall draw two more nodes C_1 and C_2 together with the lines issuing from them. Obviously, lines issuing from the nodes B should not be joined to each other, as this would give a diagram separable on one, two or three lines. Further, the lines issuing from a particular node B should not be terminated on the same node C, as this would give a diagram separable on two lines. Using the symmetry of the group of nodes A, B_1, B_2, B_3 and remembering the above remarks,

it is easily seen that there exists in all only one possibility of joining the lines, shown in Figure 20.1 by the dotted lines. The diagram obtained gives the only possible $9j$-coefficient, which admits of no further simplification.

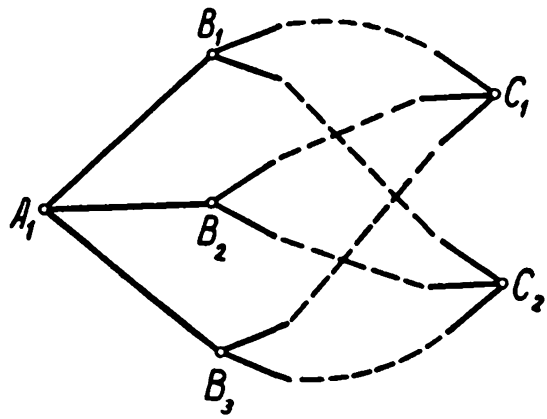

Figure 20.1

Using either method, it is easily seen that for $n=4$ there are only the two kinds of $12j$-coefficients described in the preceding section. The number of different $15j$-coefficients ($n=5$) is 5. Their diagrams, symmetry properties and algebraic expansions /Levinson and Chiplis 1958/ are given below.

The diagrams of $15j$-coefficients of the first, second, third, fourth and fifth kinds are given in Figures 20.2-20.6 respectively. As in the case of the $12j$-coefficient of the first kind, representations of the diagrams of $15j$-coefficients of the first, third and fourth kinds by plane polygons are not unique. For the sake of convenience, all such representations are given in the corresponding figures. As may be seen from Figure 20.6, the $15j$-coefficient of the fifth kind cannot be represented by a plane polygon in which the vertices cover all the nodes of the given diagram.

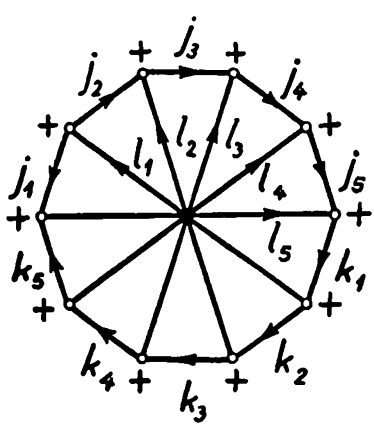

Figure 20.2a

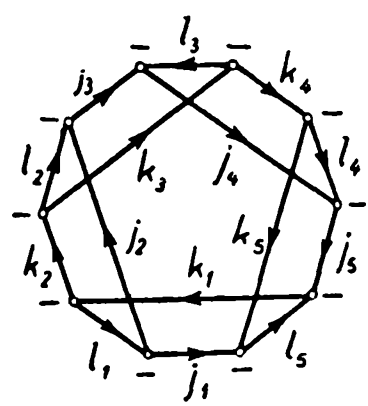

Figure 20.2b

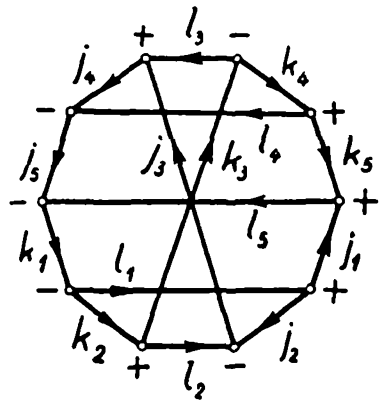

Figure 20.2c

The $15j$-coefficients of the first and second kind are particular cases of the general coefficients examined in section 17. From (17.1) and (17.2), we obtain their expressions in terms of the sums of products of $6j$-coefficients, their symmetry properties follow from (17.4) and (17.5). For the remaining types of $15j$-coefficients, the symmetry properties are established with the aid of their diagrams. These coefficients are already incapable of expression as single sums of products of $6j$-coefficients, although certain of these may be expressed as single sums in which $9j$-coefficients appear in addition to $6j$-coefficients. These expressions are obtained by the method of separating the diagrams given in section 14. We shall give these expressions and the symmetry properties for these three $15j$-coefficients.

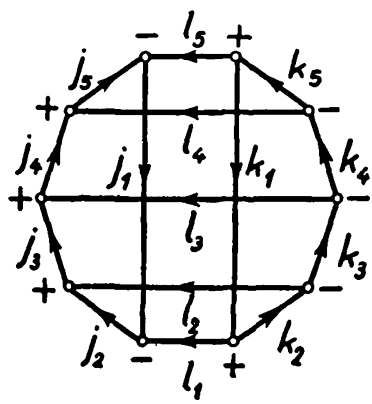

Figure 20.3

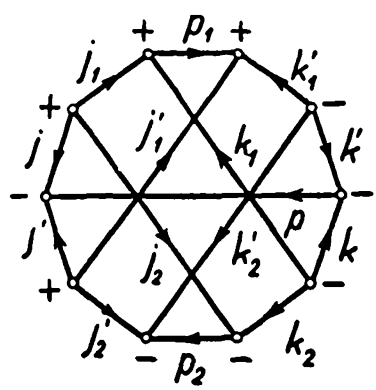

Figure 20.4a

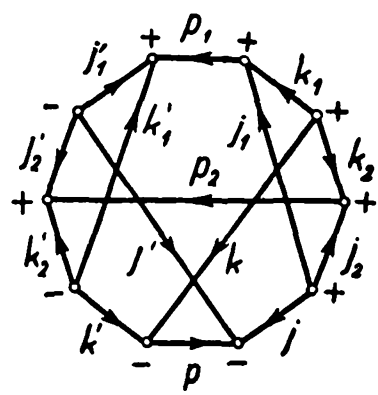

Figure 20.4b

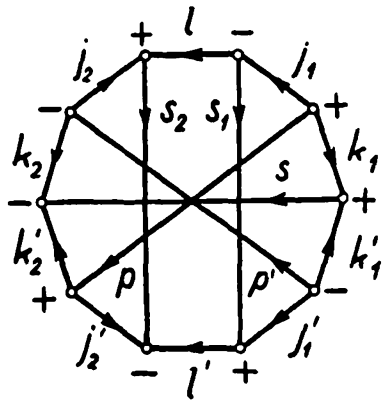

Figure 20.5a

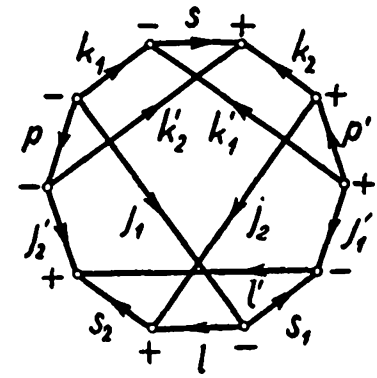

Figure 20.5b

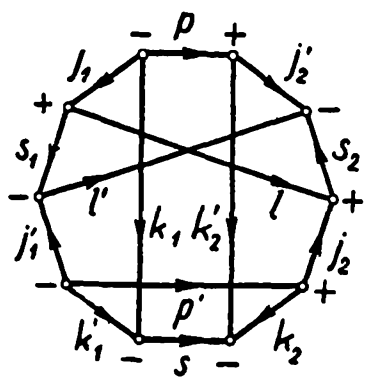

Figure 20.5c

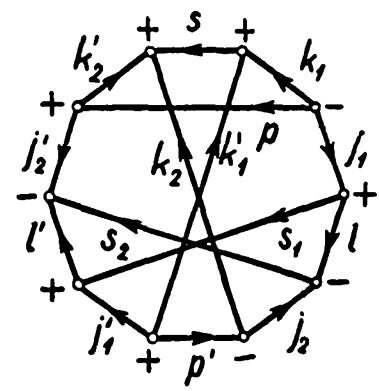

Figure 20.5d

The 15j-coefficient of the third kind possesses the following symmetry properties

$$\left\{\begin{matrix} k_1 & k_1' & k & k' & k_2 & k_2' \\ & p_1 & & p & & p_2 \\ j_1 & j_1' & j & j' & j_2 & j_2' \end{matrix}\right\} = (-1)^{\varphi_1}\left\{\begin{matrix} k_1' & k_1 & k' & k & k_2' & k_2 \\ & p_1 & & p & & p_2 \\ j_1' & j_1 & j' & j & j_2' & j_2 \end{matrix}\right\} =$$

$$= (-1)^{\varphi_2}\left\{\begin{matrix} j_1 & j_1' & j & j' & j_2 & j_2' \\ & p_1 & & p & & p_2 \\ k_1 & k_1' & k & k' & k_2 & k_2 \end{matrix}\right\} = (-1)^{\varphi_3}\left\{\begin{matrix} k_2 & k_2' & k & k' & k_1 & k_1' \\ & p_2 & & p & & p_1 \\ j_2 & j_2' & j & j' & j_1 & j_1' \end{matrix}\right\}, \quad (20.1)$$

where

$$\begin{aligned} \varphi_1 &= (j-j')-(k-k'), \\ \varphi_2 &= (j_1+j_2-j_1'-j_2')-(k_1+k_2-k_1'-k_2'), \\ \varphi_3 &= (j_1+j_2+j+j_1'+j_2'+j')- \\ &\quad -(k_1+k_2+k+k_1'+k_2'+k'), \end{aligned} \quad (20.2)$$

and is expressed as

$$\begin{Bmatrix} k_1 & k_1' & k & k' & k_2 & k_2' \\ p_1 & & p & & p_2 & \\ j_1 & j_1' & j & j' & j_2 & j_2' \end{Bmatrix} = \sum_x (x)(-1)^{x+p-j-k'} \times$$

$$\times \begin{Bmatrix} k & j & x \\ j' & k' & p \end{Bmatrix} \begin{Bmatrix} k & j & x \\ k_1 j_1 p_1 \\ k_2 j_2 p_2 \end{Bmatrix} \begin{Bmatrix} k' & j' & x \\ k_1' & j_1' & p_1 \\ k_2' & j_2' & p_2 \end{Bmatrix}. \tag{20.3}$$

The symmetry properties of the $15j$-coefficient of the fourth kind are

$$\begin{Bmatrix} j_1 & k_1 & s_1 & k_1' & j_1' \\ p & l & s & l' & p' \\ j_2 & k_2 & s_2 & k_2' & j_2' \end{Bmatrix} = (-1)^{\omega} \begin{Bmatrix} j_1' & k_1' & s_1 & k_1 & j_1 \\ p' & l' & s & l & p \\ j_2' & k_2' & s_2 & k_2 & j_2 \end{Bmatrix} =$$

$$= \begin{Bmatrix} j_2 & k_2 & s_2 & k_2' & j_2' \\ p' & l & s & l' & p \\ j_1 & k_1 & s_1 & k_1' & j_1' \end{Bmatrix}, \tag{20.4}$$

where

$$\omega = k_1 - k_2 + k_1' - k_2' + 2s_1 + 2s_2, \tag{20.5}$$

and it is given by

$$\begin{Bmatrix} j_1 & k_1 & s_1 & k_1' & j_1' \\ p & l & s & l' & p' \\ j_2 & k_2 & s_2 & k_2' & j_2' \end{Bmatrix} = (-1)^{k_1+k_2-s_1-s_2+p+p'+2l'} \times$$

$$\times \sum_x (x) \begin{Bmatrix} j_1 & j_2' & x \\ l & s_2 & j_2 \\ s_1 & l' & j_1' \end{Bmatrix} \begin{Bmatrix} j_1 & j_2' & x \\ k_2' & k_1 & p \end{Bmatrix} \begin{Bmatrix} j_2 & j_1' & x \\ k_1' & k_2 & p \end{Bmatrix} \begin{Bmatrix} k_1 & k_2' & x \\ k_2 & k_1' & s \end{Bmatrix}. \tag{20.6}$$

Finally, for the $15j$-coefficient of the fifth kind we have the symmetry properties

$$\begin{Bmatrix} k_1 & k_1' & j_1 & l_1 & l_1' \\ k_2 & k_2' & j_2 & l_2 & l_2' \\ k_3 & k_3' & j_3 & l_3 & l_3' \end{Bmatrix} = (-1)^{\eta_1} \begin{Bmatrix} k_1 & k_1' & j_1 & l_1' & l_1 \\ k_3 & k_3' & j_3 & l_3' & l_3 \\ k_2 & k_2' & j_2 & l_2' & l_2 \end{Bmatrix} = (-1)^{\eta_2} \begin{Bmatrix} k_1' & k_1 & j_1 & l_1 & l_1' \\ k_3' & k_3 & j_3 & l_3 & l_3' \\ k_2' & k_2 & j_2 & l_2 & l_2' \end{Bmatrix} =$$

$$= (-1)^{\eta_3} \begin{Bmatrix} k_2 & k_2' & j_2 & l_2 & l_2' \\ k_3 & k_3' & j_3 & l_3 & l_3' \\ k_1 & k_1' & j_1 & l_1 & l_1' \end{Bmatrix} = \begin{Bmatrix} k_2 & k_2' & j_2 & l_2 & l_2' \\ k_3 & k_3' & j_3 & l_3 & l_3' \\ k_1 & k_1' & j_1 & l_1 & l_1' \end{Bmatrix}, \tag{20.7}$$

where

$$\begin{aligned} \eta_1 &= l_1 + l_2 + l_3 - l_1' - l_2' - l_3', \\ \eta_2 &= l_1 + l_2 + l_3 + l_1' + l_2' + l_3' + j_1 + j_2 + j_3, \\ \eta_3 &= l_1 - l_2 + l_2' - k_2' + k_3 + j_2, \end{aligned} \tag{20.8}$$

and the expression

$$\begin{Bmatrix} k_1 & k_1' & j_1 & l_1 & l_1' \\ k_2 & k_2' & j_2 & l_2 & l_2' \\ k_3 & k_3' & j_3 & l_3 & l_3' \end{Bmatrix} = \sum_{x_1 x_2} (x_1)(x_2)(-1)^{j_1+j_2+k_1+k_1'-k_2+k_2'-l_2'+l_3+x_2} \times$$

$$\times \begin{Bmatrix} l_2 & l_2' & x_1 \\ k_2' & k_3' & k_1 \end{Bmatrix} \begin{Bmatrix} l_2' & l_3 & x_2 \\ k_2 & k_3 & k_1' \end{Bmatrix} \begin{Bmatrix} l_2 & l_3' & x_1 \\ l_2' & l_3 & x_2 \\ j_3 & j_2 & j_1 \end{Bmatrix} \begin{Bmatrix} k_2' & k_3' & x_1 \\ k_3 & k_2 & x_2 \\ l_1 & l_1' & j_1 \end{Bmatrix}. \qquad (20.9)$$

All the parameters in this $15j$-coefficient are equivalent. The notation for the $15j$-coefficient is taken from Levinson and Chiplis /1958/. It has been chosen for the clearest expression of the triad structure and symmetries.

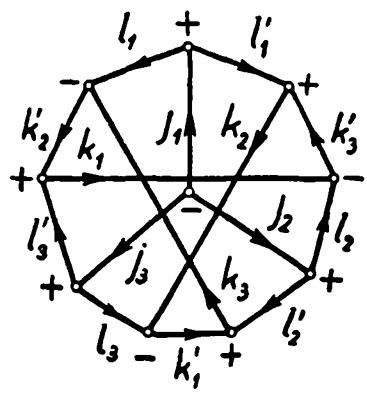

Figure 20.6

The $18j$-coefficient has been studied by Budrite and Gutman /1959/. Owing to the size of the diagrams and formulas, they have been relegated to appendices 3 and 4.

Chapter V

UTILIZATION OF TRANSFORMATION MATRICES FOR OBTAINING SUM RULES AND TRANSFORMATION FORMULAS FOR *jm*-COEFFICIENTS

In sections 8 and 9 (Chapter II) we examined the general properties of transformation matrices of eigenfunctions of coupled angular momenta. It is evident from (8.8) that the elements of the transformation matrix can be expressed in terms of the product of two generalized Clebsch-Gordan coefficients summed over all the magnetic quantum numbers. As the generalized Clebsch-Gordan coefficients can be expressed in terms of generalized Wigner coefficients, we see that the elements of the transformation matrix may be expressed as sums of products of Wigner coefficients which are independent of the magnetic quantum numbers. In other words, these matrices may be expressed in terms of j-coefficients. Herein lies the main significance of j-coefficients, and the importance of studying the relation between transformation matrices and j-coefficients. By determining this relation one can obtain the properties of j-coefficients from the properties of transformation matrices. It is to this question that the present chapter is devoted

Section 21 deals with the general consideration of the relation between transformation matrices and j-coefficients. In the next section we consider methods of separating the explicit expressions for the transformation matrix in terms of j-coefficients, section 23 gives these expressions for the case of the simplest transformation matrices. Section 24 is devoted to obtaining sum rules for products of j-coefficients from matrix identities. Finally, in section 25 we indicate a method for obtaining transformation formulas for jm-coefficients with the help of transformation matrices.

21. General considerations on the relation between transformation matrices and j-coefficients

When studying the relation between transformation matrices and j-coefficients, the transformation matrix may, without loss of generality, be characterized by a single permutation of the component angular momenta. In fact, the matrices

$$\left((P_1 j_1 \dots j_n)^A aJ \middle| (P_2 j_1 \dots j_n)^B bJ\right) \tag{21.1}$$

and

$$\left((E j_1 \dots j_n)^A aJ \middle| (P_2 P_1^{-1} j_1 \dots j_n)^B bJ\right) \tag{21.2}$$

may be expressed in terms of j-coefficients of the same type This follows from the fact that the matrix (21.2) may be obtained from (21.1) by renumbering the subscripts of the angular momenta j, which does not affect the structure of the matrix. We can

thus confine our attention to matrices of the following form

$$\left((j_1 \dots j_n)^A aJ \middle| (Pj_1 \dots j_n)^B bJ\right). \tag{21.3}$$

It is seen from here that for the given schemes of coupling A and B (see section 6), there can be $n!$ matrices characterized by permutations of the subscripts 1, 2, . . ., n in the right-hand side of (21.3). Those matrices in which the same two angular momenta are added together on both sides need not be considered, since, owing to the unitarity of the Clebsch-Gordan coefficients, they may be expressed in terms of the products of transformation matrices of the eigenfunction of $n-2$ coupled angular momenta and the Kronecker delta. There exist several elementary ways of studying the remaining matrices, these make it possible to separate sets of matrices which may be expressed in terms of j-coefficients of the same type /Vanagas and Chiplis 1958/. These methods are based on the symmetries of the Clebsch-Gordan coefficients (4.7c) and renumbering of the subscripts. We shall examine them in greater detail.

In accordance with (8.7), the matrix (21.3) and the matrix

$$\left((Pj_1 \dots j_n)^B bJ \middle| (j_1 \dots j_n)^A aJ\right) \tag{21.4}$$

are equal. We shall renumber the subscripts of j in the above matrix according to the permutation P^{-1}. We then obtain the matrix

$$\left((j_1 \dots j_n)^B bJ \middle| (P^{-1}j_1 \dots j_n)^A aJ\right). \tag{21.5}$$

From the above, the matrices (21.4) and (21.5) are expressed in terms of j-coefficients of the same type. This property is useful when $A = B$ and $P \neq P^{-1}$.

If the scheme of coupling B in the matrix (21.3) is such that the angular momenta j_t and j_k are added directly and the matrix is characterized by the permutation

$$P = \begin{pmatrix} 1 & 2 & \dots & \dots & \dots & n \\ i_1 & i_2 & \dots & t & \dots & k \dots i_n \end{pmatrix}, \tag{21.6}$$

then it follows from the symmetry properties of the Clebsch-Gordan coefficients that the matrix characterized by the permutation

$$P = \begin{pmatrix} 1 & 2 & \dots & \dots & \dots & n \\ i_1 & i_2 & \dots & k & \dots & t \dots i_n \end{pmatrix} \tag{21.7}$$

will differ from the first matrix only by a phase factor, i.e., both these matrices may be expressed in terms of j-coefficients of the same type.

If the coupling scheme A in the matrix (21.3) is such that the angular momenta j_t and j_k are added directly and the matrix is characterized by the permutation (21.6), then by interchanging these angular momenta in the left side of the matrix and renumbering the subscripts k and t so as to read t and k respectively, we will obtain a new matrix characterized by the permutation (21.7). Owing to the fact that a permutation of the coupled angular momenta can only change the phase of the matrix, and renumbering of the subscripts does not influence it structure, the original matrix and the one obtained can be expressed in terms of j-coefficients of the same type.

By using these elementary methods one can distinguish matrices which differ essentially from each other from those which do not. Thus, using the symmetry properties of the Clebsch-Gordan coefficients and renumbering the subscripts, one can bring any scheme of addition to a form in which the angular momenta j_1 and j_2 in the left-hand side and j_{i_1} and j_{i_2} in the right are added directly. Then the $2(n-2)!$ matrices for which $i_1 = 1$ and $i_2 = 2$ or $i_1 = 2$ and $i_2 = 1$ may be expressed in terms of the product of the Kronecker delta and the transformation matrix of the eigenfunctions of $n-2$ coupled angular momenta. Half of the remaining $n! - 2(n-2)!$ matrices have the same structure owing to the permutation of j_{i_1} and j_{i_2} $(i_1, i_2 \neq 1, 2)$. Further, half the remaining matrices have the same structure owing to the possibility of renumbering the subscripts 1 and 2. Thus for any coupling schemes A and B it remains to consider not more than

$$\frac{n! - 2(n-2)!}{4} \tag{21.8}$$

matrices.

If the coupling schemes are such that more than one pair of angular momenta are directly added at a time, then the number of essentially different matrices is still further reduced. For example, it is easily seen that when the coupling scheme A is such that two pairs of angular momenta B are directly coupled, for any coupling scheme j_i it is sufficient to consider

$$\frac{n! - 4(n-2)!}{8} \tag{21.9}$$

matrices.

Let us consider in greater detail the influence of the coupling scheme on the structure of the transformation matrices. For a large number of added angular momenta n, there exist many different schemes of coupling. In the simplest cases they are added in the following way. The angular momenta are divided into sets of, say, $k_1, k_2, \ldots, k_i$ angular momenta $(\Sigma k_i = n)$. In each of these sets the angular momenta are added step-by-step, namely

$$\Big(\ldots\Big(\big((\cdot+\cdot)+\cdot\big)+\cdot\Big)\ldots\Big). \tag{21.10}$$

The resultant angular momenta of these sets are added in the same manner. This coupling scheme is conveniently denoted by the symbol $A^{k_1, k_2, \ldots, k_s}$. Thus, for $n=6$, $k_1=2$, $k_2=3$ and $k_3=1$, $k_4 = k_5 = \ldots = 0$, we have

$$A^{2,3,1} = \Big(\big((\cdot+\cdot)+\big((\cdot+\cdot)+\cdot\big)\big)+\cdot\Big) \tag{21.11}$$

For any scheme of coupling B, $A^{2,3,1}$ and $A^{3,2,1}$ give matrices with identical structures. Consequently, the number of essentially different coupling schemes is substantially reduced. For small n this number is altogether insignificant. For $n=3$ there is only one scheme A^3, for $n=4$ there are two, A^4 and $A^{2,2}$, and so forth.

Henceforth we shall use the following abbreviated notation for the various coupling schemes

$$\begin{aligned} A^n &= A_0, \\ A^{2, n-2} &= A_1, \\ A^{2,2,n-4} &= A_2. \end{aligned} \tag{21.12}$$

To simplify the writing we shall introduce a somewhat condensed notation for an element of the transformation matrix. Thus, the matrix (21.3) shall be denoted as follows·

$$\left((12 \ldots n)^A \,|\, (i_1 \ldots i_n)^B\right). \tag{21.13}$$

In these symbols the component angular momenta will be indicated only by the corresponding subscripts. Intermediate and resultant angular momenta can be dropped.

From now on in this notation we shall write matrix identities of the type of

$$\left((1 \ldots n)^A \,|\, (i_1 \ldots i_n)^B\right) = \sum_{\substack{c_1 \ldots c_{n-2} \\ d_1 \ldots d_{n-2}}} \left((1 \ldots n)^A \,|\, (k_1 \ldots k_n)^C\right) \times$$

$$\times \left((k_1 \ldots k_n)^C \,|\, (p_1 \ldots p_n)^D\right)\left((p_1 \ldots p_n)^D \,|\, (i_1 \ldots i_n)^B\right), \tag{21.14}$$

in the abbreviated form

$$\left((1 \ldots n)^A \,|\, (i_1 \ldots i_n)^B\right) = \sum_{\substack{c_1 \ldots c_{n-2} \\ d_1 \ldots d_{n-2}}} \left((1 \ldots n)^A \,|\, (k_1 \ldots k_n)^C \,|\right.$$

$$\left. (p_1 \ldots p_n)^D \,|\, (i_1 \ldots i_n)^B\right). \tag{21.15}$$

Further, we shall denote the permutation (6.17) by the symbol $P_{i_1 i_2 \ldots i_n}$.

22 Methods for obtaining the relation between transformation matrices and j-coefficients

The expression for transformation matrices in terms of j-coefficients is most conveniently found by the graphical method. We therefore represent the generalized Wigner coefficients appearing in the expression for the transformation matrix in the right-hand side of (8.8) in graphical form according to the method given in section 12. Further summation over $m_1, \ldots, m_n$ entails contracting the two diagrams, this leads to a closed diagram. For practical calculations it is useful to establish a general rule, given below, for expressing the transformation matrix in terms of j-coefficients which will enable us to obtain directly the phase factor as well.

The diagrams representing the generalized Wigner coefficients should be drawn in such a way that the nodes representing the triads on the right side of the matrix have a "+" sign, lines representing the compounded angular momenta should be directed towards the node, and the line representing the resultant should be directed away from the node. Further, the lines representing the compounded angular momenta in a triad should be so placed that the "first" can be brought into coincidence with the "second" by a counter-clockwise rotation. On the left side of the matrix, the nodes should have a "−" sign and the sense of the lines and the mutual position of the first and the second line in a triad should be the reverse of the above. After drawing the nodes and the lines in this fashion, all corresponding lines should be contracted. A direct calculation will show that the transformation matrix will be

equal to the j-coefficient represented by the diagram set up in the manner indicated above, multiplied by

$$(-1)^{2(J+\sum_{i=1}^{n-2} b_i+S)} \cdot C, \tag{22.1}$$

where S is the sum of all "first" coupled angular momenta, and

$$C=\left[\prod_{i=1}^{n-2}(2a_i+1)(2b_i+1)\right]^{\frac{1}{2}}. \tag{22.2}$$

For the coupling scheme $A=B=A_0$, the phase factor may be rewritten as follows

$$(-1)^{2(J+j_1+j_{i_1}+\sum_{i=1}^{n-2} a_i)}, \tag{22.3}$$

where Σa_i is the sum of all intermediate angular momenta on the left side of the matrix.

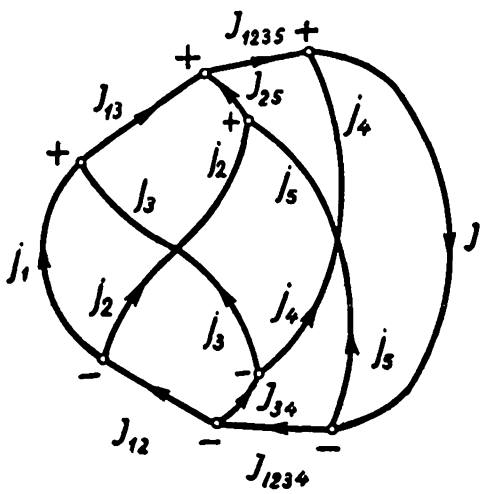

Figure 22.1a

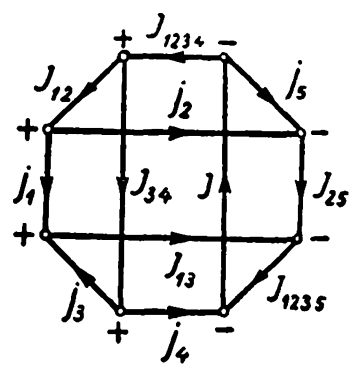

Figure 22.1b

As an example of the use of the above rules, we quote the expression

$$\left((12345)^{A_0}|(13254)^{A_0}\right)=$$

$$(-1)^{2(J+J_{12}+J_{12}+J_{1235}+j_1+j_3+J_{12}+J_{1234}+j_1+j_3+J_{12}+J_{1235})}\times$$

$$\times\left[(J_{12})(J_{34})(J_{1234})(J_{13})(J_{25})(J_{1235})\right]^{\frac{1}{2}}\cdot F, \tag{22.4}$$

where F is the j-coefficient represented by Figure 22.1a, which may be brought to

the form of Figure 22.1b Further, comparing Figures 22.1b and 19.2, we obtain

$$\left((12345)^{A_2}|(13254)^{A_2}\right)=\left(\left(\left((j_1 j_2)J_{12}(j_3 j_4)J_{34}\right)J_{1234}j_5\right)J|\right.$$

$$\left.\left(\left((j_1 j_3)J_{13}(j_2 j_5)J_{25}\right)J_{1325}j_4\right)J\right)=$$

$$=(-1)^{j+j_1-j_{12}-j_{34}}\left[(J_{12})(J_{34})(J_{1234})(J_{13})(J_{25})(J_{1325})\right]^{\frac{1}{2}}\begin{bmatrix} j_3 & J_{12} & J_{1325} & j_5 \\ J_{13} & j_4 & j_2 & J_{1234} \\ j_1 & J_{25} & J_{34} & J \end{bmatrix}. \quad (22.5)$$

Transformation matrices may also be studied by a purely algebraic method. For this it is necessary to reduce the matrix to a sum of products of simpler matrices with known expressions in terms of j-coefficients. The required expression is obtained by carrying out the summation. As an example, let us express the matrix under consideration as

$$\left((12345)^{A_2}|(13254)^{A_1}\right)=$$

$$=\sum_{J_{123}}\left((12345)^{A_2}|(12345)^{A_0}|(12354)^{A_0}|(13254)^{A_0}|(13254)^{A_2}\right). \quad (22.6)$$

All the matrices in the right-hand side may be expressed in temrs of $6j$-coefficients after performing the simplification (23.1), given in the following section. Subsequently, the expression (22.5) may be recovered by using (19.3).

It should be noted that the graphical method is the most convenient for studying the transformation matrices, as it does not involve a search for means of expressing the given matrix in terms of simpler matrices. Moreover, this expression is sometimes rather artificial in character and is not always easy to find.

23. Explicit expressions for the simplest transformation matrices

In this section we shall examine the expressions for transformation matrices for three, four and five coupled angular momenta. Of the matrices which are reducible to each other by elementary methods (as described in the preceding section) we choose the matrix which is characterized by the highest permutation in the lexicographic order.

As we noted in section 21, for three angular momenta there is only the scheme of coupling A_0. It follows from (21.8) that in this case it is sufficient to consider the matrix characterized by the permutation P_{132}. This matrix was examined for the first time by Racah /1943/. It is expressed in terms of the $6j$-coefficient as follows

$$\left((123)^{A_0}|(132)^{A_0}\right)=(-1)^{j_1+j_2+j_{12}+j_{13}}\left[(J_{12})(J_{13})\right]^{\frac{1}{2}}\begin{Bmatrix} j_2 & j_1 & J_{12} \\ j_3 & J & J_{13} \end{Bmatrix}. \quad (23.1)$$

For the addition of four angular momenta, there are two different schemes A_0 and A_1. One should therefore examine the matrices characterized by the schemes $(A_0|A_0)$, $(A_1|A_0)$ and $(A_1|A_1)$. It follows from (21.8) that there are five matrices

for the schemes $(A_0|A_0)$. The permutations which characterize these matrices are P_{1324}, P_{1342}, P_{1423}, P_{3412} and P_{1432}. The first of these may be simplified by the method of section 9. The matrix P_{1423} may be reduced to P_{1342} by renumbering the subscripts. With the aid of the expansion

$$\left((1234)^{A_0}|(1342)^{A_0}\right)=\left((1234)^{A_0}|(1324)^{A_0}|(1342)^{A_0}\right), \tag{23.2}$$

in which summation over the intermediate angular momenta drops out owing to diagonality (see the end of section 8), the latter matrix may be reduced to the product of two $6j$-coefficients. The matrix characterized by the permutation P_{3412} is expanded in a similar way

$$\left((1234)^{A_0}|(3412)^{A_0}\right)=\left((1234)^{A_0}|(1234)^{A_1}|(3412)^{A_0}\right), \tag{23.3}$$

and therefore also reduces to the product of two $6j$-coefficients. The matrix P_{1432} can be expressed in terms of the $9j$-coefficient as follows /Arima et al. 1954/

$$\left((1234)^{A_0}|(1432)^{A_0}\right)=(-1)^{j_{12}-j_{14}-j_{123}+j_{134}}\left[(J_{12})(J_{123})(J_{14})(J_{134})\right]^{\frac{1}{2}}\times$$
$$\times\begin{Bmatrix} j_1 & J_{12} & j_2 \\ J_{14} & j_3 & J_{134} \\ j_4 & J_{123} & J \end{Bmatrix}. \tag{23.4}$$

It follows from (21.8) that for the coupling schemes $(A_1|A_0)$ it is sufficient to consider the two matrices P_{1324} and P_{1432}. The two are expanded as follows

$$\left((1234)^{A_1}|(1324)^{A_0}\right)=\left((1234)^{A_1}|(1234)^{A_0}|(1324)^{A_0}\right) \tag{23.5}$$

and

$$\left((1234)^{A_1}|(1432)^{A_0}\right)=\left((1234)^{A_1}|(3412)^{A_0}|(1432)^{A_0}\right). \tag{23.6}$$

The first expansion was obtained by Biederharn /1953/. These show that both matrices can be expressed as products of $6j$-coefficients.

For the coupling schemes $(A_1|A_1)$ it is sufficient to study one of the matrices which cannot be simplified further, e.g., P_{1324}. It was examined by Wigner and an expression for it given by Jahn and Hope /1954/. It is

$$\left((1234)^{A_1}|(1324)^{A_1}\right)=\left[(J_{12})(J_{34})(J_{13})(J_{24})\right]^{\frac{1}{2}}\begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix}. \tag{23.7}$$

Thus, for four added angular momenta, only two types of all possible transformation matrices require to be expressed in terms of $9j$-coefficients, while the rest reduce to the product of two $6j$-coefficients.

Let us turn to the transformation matrices of eigenfunctions of five coupled angular momenta. Here the possible coupling schemes are A_0, A_1 and A_2. Consequently, the transformation matrices of five coupled angular momenta can be characterized by the following schemes $(A_0|A_0)$, $(A_1|A_1)$, $(A_2|A_2)$, $(A_1|A_0)$, $(A_2|A_0)$ and $(A_2|A_1)$. An analysis similar to the one given above for four angular momenta leads to the results which are given in Table 23.1. The table lists the coupling schemes and the permutations (upon the order 12345). Of the matrices

which are reducible to each other, only the lexicographically highest one is shown in the table. Thus, to obtain the matrix $(A_0|A_0)$ with the permutation P_{54231}, one must permute the angular momenta (j_5, j_4) and (j_1, j_2), interchange the subscripts 1 and 2, make use of the reality of the matrix and again renumber the subscripts 4, 5, 1, 3, 2 to 1, 2, 3, 4, 5 respectively. One then obtains the matrix with the permutation P_{35412}, which, according to the table, can be expressed as a product of $6j$- and $9j$-coefficients.

Explicit expressions as products of $6j$- and $9j$-coefficients may be found for the matrices of the first three groups in this table with the aid of the expansions (A5.1)-(A5.27) in appendix 5, which also gives expressions for the remaining matrices in terms of $12j$-coefficients ((A5.28)-(A5.43)).

In addition to the special matrices considered above, two general transformation matrices are known to be expressed in terms of $3nj$-coefficients. The matrix which can be expressed in terms of the $3nj$-coefficient of the first kind is /Levinson and Vanagas 1957/

$$\Big((1\dots k\,n\,n-1\dots k+1)^{A^{k,\,n-k}}\Big|$$

$$(n2\dots k\,1\,n-1\dots k+1)^{A^{k,\,n-k}}\Big)=$$

$$=(-1)^{j_1\dots k+j_{k+1}\dots n-j_{n2}\dots k-j_{k+1}\dots n-1\,1}\times$$

$$\times\Big[(J_{12})\cdots(J_{12\dots k})(J_{n-1n})\cdots(J_{k+1\dots n-1\,n})\times$$

$$\times(J_{n2})\cdots(J_{n2\dots k})(J_{n-1\,1})\cdots(J_{k+1\dots n-1n})\Big]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} j_1 & J_{12}\cdots\cdots J_{12\dots k} & J_{k+1\dots n-1\,n}\cdots J_{n-2\,n-1\,n} & J_{n-1\,n}\\ j_2 & j_3\cdots\quad J\quad & j_{k+1}\cdots\cdots j_{n-2} & j_{n-1}\\ j_n & J_{n2}\cdots\cdots J_{n2\dots k} & J_{k+1\dots n-1\,1}\cdots J_{n-2\,n-1\,1} & J_{n-1\,1}\end{Bmatrix}. \qquad (23.8)$$

If here $k=n-1$, then $A^{n-1,\,1}=A^n=A_0$ and (23.8) becomes

$$\Big((12\dots n)^{A_0}|(n2\dots n-1\,1)^{A_0}\Big)=(-1)^{j_{12}\dots n-1+j_n-j_{n2}\dots n-1-j_1}\times$$

$$\times\Big[(J_{12})\cdots(J_{12\dots n-1})(J_{n2})\cdots(J_{n2\dots n-1})\Big]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} j_1 & J_{12}\cdots\quad\cdots J_{12\dots n-1}\\ j_2 & j_3\cdots j_{n-1}\quad J\\ j_n & J_{n2}\cdots\quad\cdots J_{n2\dots n-1}\end{Bmatrix}. \qquad (23.9)$$

Table 23.1

Summary of results on transformation matrices of five coupled angular momenta

$(A_0 \mid A_0)$	$(A_1 \mid A_1)$	$(A_2 \mid A_2)$	$(A_1 \mid A_0)$	$(A_2 \mid A_0)$	$(A_2 \mid A_1)$	Expressed as a
13254			35412			product of two 6j-coefficients
13452 13524 34152 34512 45312	35142		13245 13425 13452 15234 15342 35124 35142	13254 13452 15234	15234	product of three 6j-coefficients
13542 14352 35412	13452		13254 13542	15342	13245	product of 6j- and 9j-coefficients
14532 15342 15432 35142	13245 13254		13524	13524 13542 15324	13254 13452	12j-coefficient of the first kind
14523		13254 13452	15324			12j-coefficient of the second kind

The $3nj$-coefficient of the second kind is related to the following matrix

$$\left((12\ldots n)^{A_0} \mid (145\ldots n23)^{A_0}\right) = (-1)^{j_2+j_3+J_{12}-J_{14\ldots n}+J_{14\ldots n_2}-J_{123}} \times$$

$$\times \left[(J_{12})\cdots(J_{12\ldots n-1})(J_{14})\cdots(J_{14\ldots n})(J_{14\cdots n2})\right]^{\frac{1}{2}} \times$$

$$\times \begin{bmatrix} j_3 & J_{123} & J_{1234} & \cdots & J_{12\ldots n-1} & J \\ J_{12} & j_4 & j_5 & \cdots & j_n & J_{14\ldots n2} \\ j_2 & j_1 & J_{14} & \cdots & J_{14\cdots n-1} & J_{14\ldots n} \end{bmatrix}. \qquad (23.10)$$

These formulas will be utilized later on.

24. Utilization of matrix identities for obtaining sum rules on j-coefficients

The connection between matrices and j-coefficients makes it possible to obtain sum rules on j-coefficients from matrix identities. To obtain the actual formulas one must first find the corresponding matrix identities and substitute j-coefficients for the matrix elements.

The simplest sum rules on the j-coefficients, the so-called orthogonality relations, are obtained from the unitarity conditions of the transformation matrices. For example, using (23.9) we obtain

$$\sum_{x_1 \dots x_n} (x_2) \dots (x_n) \begin{Bmatrix} j_1 & j_2 & \dots & j_n \\ & l_1 & l_2 \dots & l_n \\ k_1 & x_1 & \dots & x_n \end{Bmatrix} \begin{Bmatrix} j_1 & j_2' & \dots & j_n' \\ & l_1 & l_2 \dots & l_n \\ k_1 & x_2 & \dots & x_n \end{Bmatrix} =$$

$$= \delta(j_2, j_2') \dots \delta(j_n, j_n') \left[(j_2) \dots (j_n)\right]^{-1} \times$$

$$\times \{ j_1 l_1 j_2 \} \{ j_2 l_2 j_3 \} \dots \{ j_n l_n k_1 \}, \qquad (24.1)$$

where for the sake of convenience the angular momenta have not been written in the same way as in the original equation. For $n=2$, using (17.11a) we obtain from the above

$$\sum_x (x) \begin{Bmatrix} j_1 & l_1 & j_2 \\ k_1 & l_2 & x \end{Bmatrix} \begin{Bmatrix} j_1 & l_1 & j_2' \\ k_1 & l_2 & x \end{Bmatrix} = \delta(j_2, j_2') (j_2)^{-1} \{ j_1 l_1 j_2 \}. \qquad (24.2)$$

This formula was first obtained by Racah /1942/.

We note that there exist orthogonality relations for j-coefficients which may be expressed as products of simpler coefficients. Using the unitarity condition for the matrix (23.2) and formula (23.1), we obtain the following relation

$$\sum_{xy} (x)(y) \begin{Bmatrix} j_3 & j_1 & l_1 \\ j_2 & y & x \end{Bmatrix} \begin{Bmatrix} l_2 & j_2 & l_3 \\ l_1 & j_4 & y \end{Bmatrix} \begin{Bmatrix} j_3 & j_1 & l_1' \\ j_2 & y & x \end{Bmatrix} \begin{Bmatrix} l_2 & j_2 & l_3' \\ l_1' & j_4 & y \end{Bmatrix} =$$

$$= \delta(l_1, l_1') \delta(l_3, l_3') \left[(l_1)(l_3)\right]^{-1} \{ j_1 j_3 l_1 \} \{ l_2 l_3 j_2 \} \{ l_1 l_3 j_4 \}. \qquad (24.3)$$

However, this formula will not give a new sum rule for the $6j$-coefficients, as it may be obtained by applying (24.2) twice.

Making use of the matrix identity

$$\left((12 \dots n-1\, n)^{A_0} | (n2 \dots n-1\, 1)^{A_0}\right) =$$

$$= \sum \left(12 \dots k-2\, k-1\, k \dots n-1\, n)^{A_0}\right|$$

$$\left.(k-1\ 2 \dots k-2\ 1\, k \dots n-1\, n)^{A_0}\right|$$

$$\left.(k-1\ 2 \dots k-2\, n\ k \dots n-1\ 1)^{A_0}\right| \left.(n2 \dots n-1\, 1)^{A_0}\right) \qquad (24.4)$$

and (23.8), we obtain the following sum rule for $3nj$-coefficients of the first kind

$$(-1)^{j_{k-1}-k_{k-1}-j_k+k_k}\sum_{x_1\ldots x_{k-1}}(x_2)\ldots(x_{k-1})\times$$

$$\times\begin{Bmatrix} j_1 & j_2 & \cdots & j_{k-2} & j_{k-1} \\ & l_1 & l_2 & \cdots & l_{k-2} & j_k \\ l_{k-1} & x_2 & & \cdots & x_{k-2} & x_{k-1} \end{Bmatrix}\times$$

$$\times\begin{Bmatrix} k_1 & k_2 & \cdots & k_{k-2} & k_{k-1} \\ & l_1 & l_2 & \cdots & l_{k-2} & k_k \\ l_{k-1} & x_2 & & \cdots & x_{k-2} & x_{k-1} \end{Bmatrix}\times$$

$$\times\begin{Bmatrix} j_1 & j_k & \cdots & j_n \\ x_{k-1} & l_k & \cdots & l_n \\ k_1 & k_k & \cdots & k_n \end{Bmatrix}=\begin{Bmatrix} j_1 & \cdots & j_k & \cdots & j_n \\ & l_1 & \cdots & l_k & \cdots & l_n \\ k_1 & & \cdots & k_k & \cdots & k_n \end{Bmatrix}. \qquad (24.5)$$

Introducing $l_k=l_{k+1}=\ldots=l_n=0$ in the above expression and using (17.6), we obtain

$$(-1)^{k_n-j_n+2j_1}\sum_{x_1\ldots x_n}(x_2)\ldots(x_n)(-1)^{x_n}\times$$

$$\times\begin{Bmatrix} j_1 & j_2 & \cdots & j_{n-1} & j_n \\ & l_1 & l_2 & \cdots & l_{n-1} & k_1 \\ l_n & x_2 & & \cdots & x_{n-1} & x_n \end{Bmatrix}\times$$

$$\times\begin{Bmatrix} k_1 & k_2 & \cdots & k_{n-1} & k_n \\ & l_1 & l_2 & \cdots & l_{n-1} & j_1 \\ l_n & x_2 & & \cdots & x_{n-1} & x_n \end{Bmatrix}=\begin{Bmatrix} j_1 & \cdots & j_n \\ & l_1 & \cdots & l_n \\ k_1 & & \cdots & k_n \end{Bmatrix}. \qquad (24.6)$$

This formula expresses the $(n-1)$-fold sum of the product of two $3nj$-coefficients as a single $3nj$-coefficient. For $n=2$, using (17.11a) we obtain from (24.6) the following sum rule for $6j$-coefficients /Racah 1942/

$$\sum_x(-1)^{x+k_1+k_2}(x)\begin{Bmatrix} j_1 & j_2 & k_1 \\ l_1 & l_2 & x \end{Bmatrix}\begin{Bmatrix} j_1 & l_1 & k_2 \\ j_2 & l_2 & x \end{Bmatrix}=\begin{Bmatrix} j_1 & j_2 & k_1 \\ l_2 & l_1 & k_2 \end{Bmatrix}. \qquad (24.7)$$

where the parameters have been changed. Another example of a particular case of (24.5) is the following recurrence formula

$$(-1)^{j_1-k_1-j_{n-1}+k_{n-1}}\sum_x(x)\times\begin{Bmatrix} j_1 & \cdots & j_{n-2} & j_{n-1} \\ & l_1 & \cdots & l_{n-2} & x \\ k_1 & & \cdots & k_{n-2} & k_{n-1} \end{Bmatrix}\times$$

$$\times\begin{Bmatrix} j_1 & k_{n-1} & x \\ l_{n-1} & l_n & k_n \end{Bmatrix}\begin{Bmatrix} k_1 & j_{n-1} & x \\ l_{n-1} & l_n & j_n \end{Bmatrix}=\begin{Bmatrix} j_1 & \cdots & j_n \\ & l_1 & \cdots & l_n \\ k_1 & & \cdots & k_n \end{Bmatrix}. \qquad (24.8)$$

If here $n = 4$ and $j_4 = 0$, then, taking (17.7) into account, we obtain the following useful sum rule for $6j$-coefficients* /Biedenharn 1953/

$$\sum_x (-1)^{x+R_3}(x) \begin{Bmatrix} j_1 & j_2 & k_3 \\ l_2 & l_1 & x \end{Bmatrix} \begin{Bmatrix} j_2 & j_3 & k_1 \\ l_3 & l_2 & x \end{Bmatrix} \begin{Bmatrix} j_3 & j_1 & k_2 \\ l_1 & l_3 & x \end{Bmatrix} =$$

$$= \begin{Bmatrix} j_1 & k_2 & j_3 \\ k_1 & j_2 & k_3 \end{Bmatrix} \begin{Bmatrix} l_1 & k_2 & l_3 \\ k_1 & l_2 & k_3 \end{Bmatrix}, \quad R_3 = \sum_{i=1}^{3} (j_i + k_i + l_i). \tag{24.9}$$

The important sum rules for j-coefficients are those which make it possible to express the sum of a product of j-coefficients as products of other j-coefficients. A typical example of such a sum rule is

$$(-1)^{j_k - j_{k-1} - k_k} \sum_{x_1 \ldots x_{k-1}} (-1)^{x_{k-1}}(x_2) \ldots (x_{k-1}) \times$$

$$\times \begin{Bmatrix} k_k & k_{k+1} & \cdots & k_n & j_1 & j_2 & \cdots & j_{k-1} \\ l_k & l_{k+1} & \cdots & l_n & l_1 & l_2 & \cdots & l_{k-1} \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} j_k & j_{k+1} & \cdots & j_n & k_1 & x_2 & \cdots & x_{k-1} \\ l_{k-1} & p_2 & \cdots & p_{k-2} & p_{k-1} & & & \\ l_1 & l_2 & \cdots & l_{k-2} & k_k & & & \\ k_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & & & \end{Bmatrix} =$$

$$= \begin{Bmatrix} j_1 & j_2 & \cdots & j_{k-1} & j_{k+1} \\ l_1 & l_2 & \cdots & l_{k-2} & j_k \\ l_{k-1} & p_2 & \cdots & p_{k-2} & p_{k-1} \end{Bmatrix} \begin{Bmatrix} j_1 & j_k & \cdots & j_n \\ p_{k-1} & l_k & \cdots & l_n \\ k_1 & k_k & \cdots & k_n \end{Bmatrix}. \tag{24.10}$$

This formula may be obtained from the corresponding matrix identity or by multiplying (24.5) by

$$\begin{Bmatrix} j_1 & j_2 & \cdots & j_{k-2} & j_{k-1} \\ l_1 & l_2 & \cdots & l_{k-2} & j_k \\ l_{k-1} & k_2' & \cdots & k_{k-2}' & k_{k-1}' \end{Bmatrix}$$

and summing over $j_2 \ldots j_k$ with the aid of (24.1). The advantage of this formula is that the number of parameters in the first $3nj$-coefficient under the summation sign is not particularized.

We might mention yet another type of sum rule, which expresses a $3nj$-coefficient of the second kind as a single sum of a 3 $(n+1)j$-coefficient of the first kind. Using (17.2) and the particular case of (24.7) with $k_2 = 0$, we obtain

$$\begin{bmatrix} j_1 & \cdots & j_n \\ l_1 & \cdots & l_n \\ k_1 & \cdots & k_n \end{bmatrix} = \sum_x (x)(-1)^{x-j_1-k_1} \begin{Bmatrix} j_1 & \cdots & j_n & j_1 \\ l_1 & \cdots & l_n & x \\ k_1 & \cdots & k_n & k_1 \end{Bmatrix}. \tag{24.11}$$

---

* Known as the Biedenharn identity–Translator.

The formula for a $3nj$-coefficient of the first kind as a sum of $3(n+1)j$-coefficients of the second kind is easily obtained in a similar way.

In this section we confined ourselves to certain characteristic sum rules for j-coefficients. By combining these or by using various matrix identities and the connection between matrices and j-coefficients, one can obtain a variety of different sum rules. The most important of these are given in appendix 6, which for convenience also lists the sum rules for $6j$-, $9j$- and $12j$-coefficients given in this and in other sections.

25. Use of matrix identities for the transformation of jm-coefficients

Transformation matrices can be used to obtain formulas for the transformation of jm-coefficients. The derivation of these formulas is based on the relation (8.6), which expresses an element of the transformation matrix as a sum of products of generalized Clebsch-Gordan coefficients. Owing to the unitarity of the matrix of generalized Clebsch-Gordan coefficients, we can write this relation as

$$\left(j_1 m_1 \dots j_n m_n | (j_1 \dots j_n)^A a J M\right) = \sum_b \left(j_1 m_1 \dots j_n m_n | (j_1 \dots j_n)^B b J M\right) \times$$

$$\times \left((j_1 \dots j_n)^B b J | (j_1 \dots j_n)^A a J\right). \qquad (25\ 1)$$

Relations between jm-coefficients can be obtained from this formula, together with the formulas for the expansion of transformation matrices as sums of products, provided the transformation matrices and Clebsch-Gordan coefficients are replaced by j-coefficients and Wigner coefficients respectively. Obviously, the structure of these formulas will also depend on the schemes of coupling of the angular momenta. We shall consider some examples of such formulas.

Let the transformation matrix which appears in (25.1) be similar to (23.9). In this case (introducing a more convenient notation for the parameters) one readily obtains from (25.1) the following formula for the transformation of jm-coefficients

$$\sum_{m_2 \dots m_n} (-1)^{\sum_{i=2}^{n} (j_i - m_i)} \begin{pmatrix} j_1 & l_1 & j_2 \\ m_1 & n_1 & m_2 \end{pmatrix} \begin{pmatrix} j_2 & l_2 & j_3 \\ -m_2 & n_2 & m_3 \end{pmatrix} \dots \begin{pmatrix} j_{n-1} & l_{n-1} & j_n \\ -m_{n-1} & n_{n-1} & m_n \end{pmatrix} \times$$

$$\times \begin{pmatrix} j_n & l_n & k_1 \\ -m_n & n_n & q_1 \end{pmatrix} = \sum_{k_2 q_2 \dots k_n q_n} (-1)^{2j_1 + 2k_1 + \sum_{i=2}^{n} (k_i - q_i)} (k_2) \dots (k_n) \times$$

$$\times \begin{Bmatrix} j_1 & \dots & j_n \\ l_1 & \dots & l_n \\ k_1 & \dots & k_n \end{Bmatrix} \begin{pmatrix} k_1 & l_1 & k_2 \\ q_1 & n_1 & q_2 \end{pmatrix} \begin{pmatrix} k_2 & l_2 & k_3 \\ -q_2 & n_2 & q_3 \end{pmatrix} \dots \times$$

$$\times \dots \begin{pmatrix} k_{n-1} & l_{n-1} & k_n \\ -q_{n-1} & n_{n-1} & q_n \end{pmatrix} \begin{pmatrix} k_n & l_n & j_1 \\ -q_n & n_n & m_1 \end{pmatrix}. \qquad (25.2)$$

Multiplying the above by

$$(-1)^{l_n-n_n+j_1-m_1}\begin{pmatrix} k'_n & l_n & j_1 \\ q'_n & -n_n & -m_1 \end{pmatrix},$$

summing over n_n and m_1 and using the orthogonality property of the Wigner coefficients, we obtain the following sum rule for jm-coefficients

$$\sum_{m_1 \dots m_n n_n} (-1)^{\sum_{i=1}^{n}(j_i-m_i)+l_n-n_n} \begin{pmatrix} j_1 & l_1 & j_2 \\ m_1 & n_1 & m_2 \end{pmatrix} \times$$

$$\times \begin{pmatrix} j_2 & l_2 & j_3 \\ -m_2 & n_2 & m_3 \end{pmatrix} \dots \begin{pmatrix} j_{n-1} & l_{n-1} & j_n \\ m_{n-1} & n_{n-1} & m_n \end{pmatrix} \times$$

$$\times \begin{pmatrix} j_n & l_n & k_1 \\ -m_n & n_n & q_1 \end{pmatrix} \begin{pmatrix} k_n & l_n & j_1 \\ q_n & -n_n & -m_1 \end{pmatrix} =$$

$$= \sum_{k_2 q_2 \dots k_{n-1} q_{n-1}} (k_2) \dots (k_{n-1})(-1)^{\sum_{i=2}^{n-1}(k_i-q_i)+2j_n} \times$$

$$\times \begin{Bmatrix} j_1 & \dots & j_n \\ l_1 & \dots & l_n \\ k_1 & \dots & k_n \end{Bmatrix} \begin{pmatrix} k_1 & l_1 & k_2 \\ q_1 & n_1 & q_2 \end{pmatrix} \begin{pmatrix} k_2 & l_2 & k_3 \\ -q_2 & n_2 & q_3 \end{pmatrix} \times \dots$$

$$\dots \times \begin{pmatrix} k_{n-1} & l_{n-1} & k_n \\ -q_{n-1} & n_{n-1} & q_n \end{pmatrix}. \qquad (25.3)$$

For $n = 2$, taking (17.11a) into account, we obtain a frequently encountered transformation formula

$$\sum_{m_1 m_2 n_2} (-1)^{j_1-m_1+j_2-m_2+l_2-n_2} \begin{pmatrix} j_1 & l_1 & j_2 \\ m_1 & n_1 & -m_2 \end{pmatrix} \begin{pmatrix} j_2 & l_2 & k_1 \\ m_2 & n_2 & q_1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} k_2 & l_2 & j_1 \\ q_2 & -n_2 & -m_1 \end{pmatrix} = (-1)^{j_1+j_2+k_1+k_2} \begin{Bmatrix} j_1 & j_2 & l_1 \\ k_1 & k_2 & l_2 \end{Bmatrix} \begin{pmatrix} k_1 & l_1 & k_2 \\ q_1 & n_1 & q_2 \end{pmatrix}, \qquad (25.4)$$

which will be obtained in section 26 as an illustration of the graphical method.

For $n = 3$ we have, from (25.3) and (18.1),

$$\sum_{m_1 m_2 m_3 n_3} (-1)^{j_1-m_1+j_2-m_2+j_3-m_3+l_3-n_3} \begin{pmatrix} j_1 & l_1 & j_2 \\ m_1 & n_1 & -m_2 \end{pmatrix} \times$$

$$\times \begin{pmatrix} j_2 & l_2 & j_3 \\ m_2 & n_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_3 & l_3 & k_1 \\ m_3 & n_3 & q_1 \end{pmatrix} \begin{pmatrix} k_3 & l_3 & j_1 \\ q_3 & -n_3 & -m_1 \end{pmatrix} =$$

$$= \sum_{k_2 q_2} (k_2)(-1)^{k_2-q_2+2j_2+2k_2} \begin{Bmatrix} j_1 & l_1 & j_2 \\ k_3 & k_2 & l_2 \\ l_3 & k_1 & j_3 \end{Bmatrix} \times$$

$$\times \begin{pmatrix} k_1 & l_1 & k_2 \\ q_1 & n_1 & -q_2 \end{pmatrix} \begin{pmatrix} k_2 & l_2 & k_3 \\ q_2 & n_2 & q_3 \end{pmatrix}. \qquad (25.5)$$

Using the symmetry properties of the $9j$-coefficient and (18.2), we can express the right-hand side of the above equation as follows

$$\sum_{k_2 q_2 x}(k_2)(x)(-1)^{j_1-l_2+l_3-j_1+j_2+j_3+k_1+k_2+k_3}\begin{Bmatrix} l_3 & k_1 & j_3 \\ l_2 & j_2 & x \end{Bmatrix}\begin{Bmatrix} k_1 & l_1 & k_2 \\ k_3 & l_2 & x \end{Bmatrix}\times$$

$$\times\begin{Bmatrix} l_1 & j_2 & j_1 \\ l_3 & k_3 & x \end{Bmatrix}\begin{pmatrix} k_1 & l_1 & k_2 \\ q_1 & n_1 & -q_2 \end{pmatrix}\begin{pmatrix} k_2 & l_2 & k_3 \\ q_2 & n_2 & q_3 \end{pmatrix}. \qquad (25.6)$$

Transforming the sum over k_2 with the aid of (25.2) for $n = 2$, we obtain

$$\sum_{k_2 q_2}(k_2)(-1)^{k_2-q_2+2j_2+2k_2}\begin{Bmatrix} j_1 & l_1 & j_2 \\ k_3 & k_2 & l_2 \\ l_3 & k_1 & j_3 \end{Bmatrix}\times\begin{pmatrix} k_1 & l_1 & k_2 \\ q_1 & n_1 & -q_2 \end{pmatrix}\begin{pmatrix} k_2 & l_2 & k_3 \\ q_2 & n_2 & q_3 \end{pmatrix}=$$

$$=\sum_{xm}(x)(-1)^{j_1+j_2+j_3+k_1-l_1-l_3+x-m}\begin{Bmatrix} l_3 & k_1 & j_3 \\ l_2 & j_2 & x \end{Bmatrix}\begin{Bmatrix} l_1 & j_2 & j_1 \\ l_3 & k_3 & x \end{Bmatrix}\times$$

$$\times\begin{pmatrix} k_1 & l_2 & x \\ q_1 & n_2 & -m \end{pmatrix}\begin{pmatrix} k_3 & l_1 & x \\ q_3 & n_1 & m \end{pmatrix}. \qquad (25.7)$$

The jm-coefficients considered above possess certain symmetry properties which can easily be examined by the graphical method.

Special notations are sometimes used for a few frequently encountered jm-coefficients. For example, Simon /1953/ uses the following expression as an independent quantity

$$G_m\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & \\ j_6 & j_7 & j_8 \end{Bmatrix}=(-1)^{-j_2+j_4-j_5+j_7+\frac{j_2+j_7}{2}}\times$$

$$\times\left[(j_1)(j_2)(j_3)(j_4)(j_5)(j_6)(j_7)(j_8)\right]^{\frac{1}{2}}\times$$

$$\times\sum_x(x)\begin{pmatrix} j_2 & j_7 & x \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} j_5 & j_4 & x \\ m & -m & 0 \end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & x & j_5 \\ j_6 & j_7 & j_8 \end{Bmatrix}=$$

$$=(-1)^{j_1+j_2+m+\frac{j_2+j_7}{2}}\left[(j_1)\dots(j_8)\right]^{\frac{1}{2}}\times$$

$$\times\sum_y(y)\begin{Bmatrix} j_3 & j_7 & j_1 \\ y & j_8 & j_5 \end{Bmatrix}\begin{Bmatrix} j_6 & j_7 & j_8 \\ y & j_1 & j_4 \end{Bmatrix}\begin{pmatrix} j_4 & j_7 & y \\ m & 0 & -m \end{pmatrix}\begin{pmatrix} j_2 & j_5 & y \\ 0 & m & -m \end{pmatrix}. \qquad (25.8)$$

which possesses the symmetry

$$G_m\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & \\ j_6 & j_7 & j_8 \end{pmatrix}=(-1)^{\varphi}G_m\begin{pmatrix} j_6 & j_7 & j_8 \\ j_4 & j_5 & \\ j_1 & j_2 & j_3 \end{pmatrix}=(-1)^{\varphi}G_m\begin{pmatrix} j_3 & j_2 & j_1 \\ j_5 & j_4 & \\ j_8 & j_7 & j_6 \end{pmatrix}, \qquad (25.9)$$

where $\varphi=j_1+j_3+j_4+j_5+j_6+j_8$. The quantity G_m depends on eight angular

momentum parameters. It is therefore often called the $8j$-coefficient. This name, however, is not entirely successful as G_m is essentially a jm-coefficient in which certain magnetic quantum numbers are zero

The examples of the method of obtaining transformation formulas for jm-coefficients given in this section were intended merely as illustrations. In practical calculations one meets with highly varied sums of products of Wigner coefficients and in the majority of cases the corresponding calculations are more conveniently carried out by the graphical method which was developed in Chapter III. Examples of such calculations will be given in the following chapter.

Chapter VI

EXAMPLES OF APPLICATION OF THE GRAPHICAL METHOD

In practical calculations both the summation of products of Wigner coefficients and j-coefficients and the simplification of these sums are of importance. In the two preceding sections we have indicated how this may be done with the help of matrix algebra. In the present chapter we shall deal with the use of the graphical method of this purpose. As the graphical method is convenient for carrying out the summations, we considered it advisable to give a number of examples of the practical application of this method.

In section 26 we give the general method of graphical summation of a product of Wigner coefficients over magnetic quantum numbers, together with simple examples. In the next section we analyze a more complex case. Section 28 is devoted to the summation of products of j-coefficients. In section 29 we consider an example of summation of a product of j-coefficients and a jm-coefficient. At the end of the section is sketched a practical prescription for the simultaneous evaluation of a multiple sum over angular momentum parameters.

Finally, section 30 deals with the question of choosing a method of calculation. The expediency of using two different methods of calculation is indicated and simple examples are given in which the graphical method is used to perform the summation and the phase factor is determined algebraically.

26. Graphical summation of products of Wigner coefficients

In this section we examine sums of products of Wigner coefficients in which j-coefficients do not appear explicitly. As we know, these sums arise from sums of products of Clebsch-Gordan coefficients, which do not contain transformation matrices in an explicit form. Our purpose in this section is to clarify the graphical method of summation by means of examples.

If the sum under consideration is a j-coefficient, i.e., is summed over all magnetic quantum numbers, then in order to express the summation one must draw the corresponding diagram according to the method discussed in section 12. If the diagram obtained is not separable on less than four lines, it must be compared, after being brought to the form of a polygon, with the (conventionally) standard diagrams for the given number of parameters. The case of six or nine parameters will give the diagrams 16.1 or 18.1. Reduction of the directions of lines and of the signs (using the rules in section 12) to those in the above diagrams will give the phase factor of the required expression. For 12 parameters we will obtain one of the two diagrams in Figure 19.1 (for a 12 j-coefficient of the first kind) or the diagram in Figure 19.2

(for a 12 j-coefficient of the second kind). For 15 parameters the diagram obtained will be identical with one of the diagrams in section 20. The diagrams of 18 j-coefficients in appendix 3 give only one possibility for each 18 j-coefficient, without going into all possible forms. Consequently, if a diagram obtained fails to coincide with any of these diagrams, it should be re-drawn in another form.

Should one obtain a diagram separable on less than four lines, it should first be separated by the method indicated in section 14, the separate parts should then be identified in the manner indicated above.

When the sum under consideration is a jm-coefficient, one must make use of theorem (13.5) for the expansion of a jm-coefficient in generalized Wigner coefficients. When using this theorem for actual calculations, one should draw a diagram representing the sum under consideration, choose a generalized Wigner coefficient and contract the two diagrams. The directions of free lines should be chosen appropriately in advance. The diagram obtained will represent a j-coefficient which is identified in the above manner and which will be the expansion coefficient in (13.5). To illustrate the method, we shall consider simple instances of these sums.

Let us first consider the sum

$$F\begin{pmatrix} l_1 l_2 l_3, & j_1 & j_2 & j_3 \\ & m_1 & m_2 & m_3 \end{pmatrix} = \sum_{n_1 n_2 n_3} (-1)^{l_1 - n_1 + l_2 - n_2 + l_3 - n_3} \times$$

$$\times \begin{pmatrix} l_1 & j_1 & l_2 \\ -n_1 & m_1 & n_2 \end{pmatrix} \begin{pmatrix} l_2 & j_2 & l_3 \\ -n_2 & m_2 & n_3 \end{pmatrix} \begin{pmatrix} l_3 & j_3 & l_1 \\ -n_3 & m_3 & n_1 \end{pmatrix}. \qquad (26.1)$$

The diagram of this jm-coefficient (Figure 26.1) is the simplest nontrivial jm-coefficient (the trivial case would have been the usual Wigner coefficient) with three free

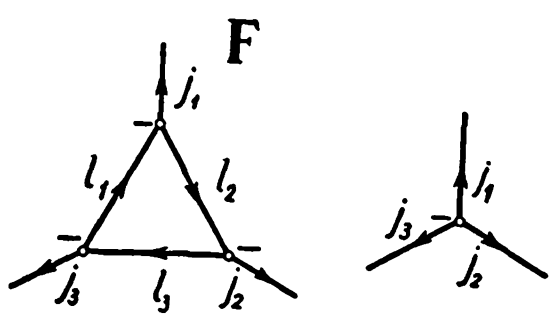

Figure 26.1

lines. In the given case the chosen generalized Wigner coefficient is a usual Wigner coefficient with the angular momenta j_1, j_2, j_3, the graphical representation of which shows the lines issuing from a node. Changing the directions of the lines in the diagram of the Wigner coefficient, contracting it with the diagram of the jm-coefficient and comparing the diagram obtained with Figure 16.1, we obtain the following

formula

$$\sum_{m_1 m_2 m_3} (-1)^{l_1-m_1+l_2-m_2+l_3-m_3} \times \begin{pmatrix} l_1 & j_1 & l_2 \\ n_1 & m_1 & n_2 \end{pmatrix} \begin{pmatrix} l_2 & j_2 & l_3 \\ -n_2 & m_2 & n_3 \end{pmatrix} \begin{pmatrix} l_3 & j_3 & l_1 \\ -n_3 & m_3 & -n_1 \end{pmatrix} =$$

$$= (-1)^{j_1-j_2+j_3} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & l_1 & l_2 \end{Bmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \qquad (26.2)$$

which is a somewhat different form of (25.4).

Let us now analyse in detail the sum (11.3), the graphical representation of which is given in Figure 26.2a. It is the simplest jm-coefficient with four free lines,

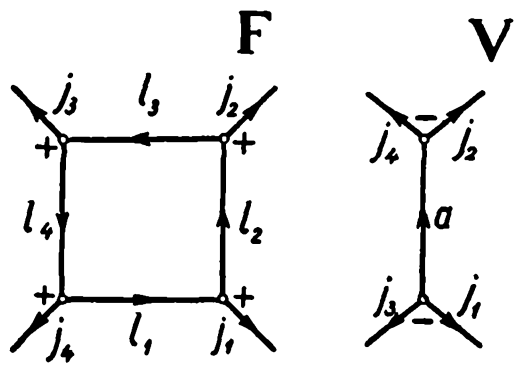

Figure 26.2a

provided we disregard the trivial case of the generalized Wigner coefficient with four angular momenta (three component and one resultant).

Let us choose the generalized Wigner coefficient given by diagram V in Figure 26.2a. Joining the lines j_1, j_2, j_3 and j_4 in this diagram to the corresponding lines in diagram F (which retain their directions), we obtain diagram R, which

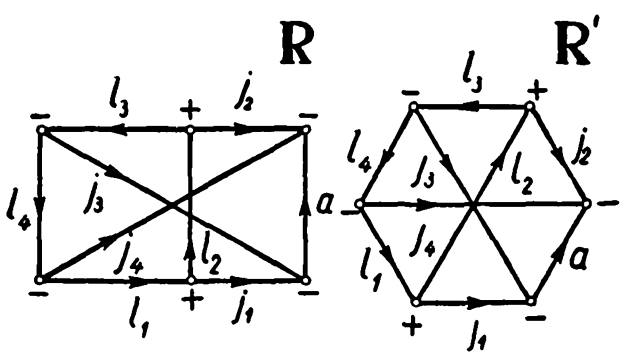

Figure 26.2b

can be represented as R' (Figure 26.2b). Comparing R' with the diagram of the

$9j$-coefficient in Figure 18.1, we obtain

$$R = R' = (-1)^{j_3+j_4-l_3-l_4-a} \begin{Bmatrix} l_4 & l_3 & j_3 \\ l_1 & l_2 & j_1 \\ j_4 & j_2 & a \end{Bmatrix}. \tag{26.3}$$

Thus from (13.5) we finally have

$$\sum_{n_1 n_2 n_3 n_4} (-1)^{l_1-n_1+l_2-n_2+l_3-n_3+l_4-n_4} \times$$

$$\times \begin{pmatrix} l_1 & j_1 & l_2 \\ -n_1 & m_1 & n_2 \end{pmatrix} \begin{pmatrix} l_2 & j_2 & l_3 \\ -n_2 & m_2 & n_3 \end{pmatrix} \begin{pmatrix} l_3 & j_3 & l_4 \\ -n_3 & m_3 & n_4 \end{pmatrix} \begin{pmatrix} l_4 & j_4 & l_1 \\ -n_4 & m_4 & n_1 \end{pmatrix} =$$

$$= \sum_{a\mu} (a)(-1)^{a-\mu} \cdot (-1)^{j_3+j_4-l_2-l_4-a} \begin{Bmatrix} l_4 & l_3 & j_3 \\ l_1 & l_2 & j_1 \\ j_4 & j_2 & a \end{Bmatrix} \times$$

$$\times \begin{pmatrix} j_1 & j_3 & a \\ m_1 & m_3 & \mu \end{pmatrix} \begin{pmatrix} a & j_4 & j_2 \\ -\mu & m_4 & m_2 \end{pmatrix}. \tag{26.4}$$

Another advantage of the general formula (13.5) is that by changing the appearance of the generalized Wigner coefficient in it one can change the j-coefficient R,

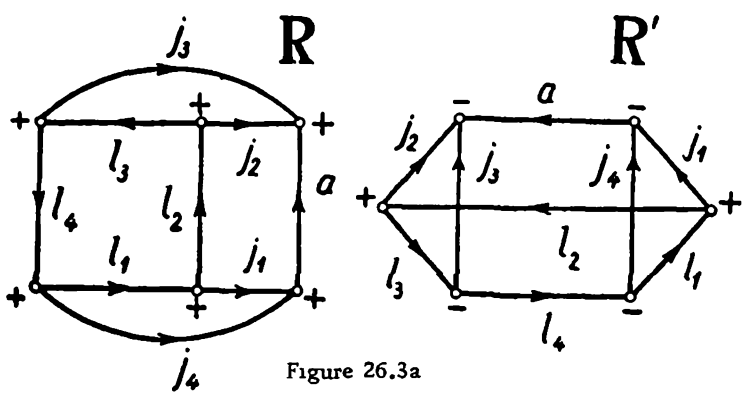

Figure 26.3a

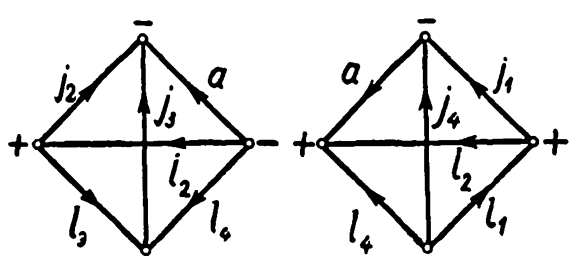

Figure 26.3b

choosing it in the simplest or most convenient form. Thus by interchanging the parameters j_3 and j_4 in the generalized Wigner coefficient in our example, we obtain the j-coefficient R in the form represented in Figure 26.3a. Let us re-draw it in the

form R', change the direction of the line l_4 and separate R' on a, l_2, l_4, using (14.5). We then find

$$R = R' = (-1)^{2l_4} R_1 \cdot R_2 \tag{26.5}$$

(R_1 and R_2 are shown in Figure 26.3b). Comparing the diagrams of R_1 and R_2 with the diagram of the $6j$-coefficient in Figure 16.1, we obtain finally

$$F\begin{pmatrix} l_1 l_2 l_3 l_4; & j_1 & j_2 & j_3 & j_4 \\ & m_1 & m_2 & m_3 & m_4 \end{pmatrix} = \sum_{a\mu} (a) \cdot (-1)^{a-\mu} \cdot (-1)^{2j_4+2j_3+l_3-l_4-a} \times$$

$$\times \begin{Bmatrix} l_4 & l_2 & a \\ j_2 & j_3 & l_3 \end{Bmatrix} \begin{Bmatrix} l_4 & l_3 & a \\ j_1 & j_4 & l_1 \end{Bmatrix} \begin{pmatrix} j_1 & j_4 & a \\ m_1 & m_4 & \mu \end{pmatrix} \begin{pmatrix} a & j_3 & j_2 \\ -\mu & m_3 & m_2 \end{pmatrix} \tag{26.6}$$

This last expression is a different form of the right-hand side of (26.4). Obtaining these formulas by the graphical method is considerably simpler than the derivations given earlier in section 25.

27. A more complex product of Wigner coefficients

In the preceding section we discussed simple examples of the summation of products of Wigner coefficients. In this section we will examine a more complex example in which there are five magnetic quantum numbers which are not summed over.

Consider the following sum of a product of seven Clebsch-Gordan coefficients

$$\sum (-1)^{\mu_1} (ks_2 q m_2 \mid ks_2 s_1 m_1)(k' s_2' s_1' m_1' \mid k' s_2' q' m_2') \times$$

$$\times (l_1' s_1' J_1 M_1 \mid l_1' s_1' \mu_1' m_1')(l_1 s_1 \mu_1 m_1 \mid l_1 s_1 J_1 M_1) \times$$

$$\times (l_2 s_2 J_2 M_2 \mid l_2 s_2 \mu_2 m_2)(l_2 s_2' \mu_2' m_2' \mid l_2' s_2' J_2 M_2) \times$$

$$\times (l_1' l_2' l' \mu' \mid l_1' l_2' - \mu_1' \mu_2') \tag{27.1}$$

(the summation is over m_1, m_2, m_1', m_2', μ_1', μ_2', M_1, M_2) which occurs in /Simon 1953/. The sum (27.1) is somewhat different from the sum (2.9) in the latter work, as it is written in the form which would result if one always placed the complex conjugate functions in the left-hand side of the matrix element in the calculations. Further, we have retained all the summations and have not particularized the values of μ_1 and μ_2, which are usually zero, and of μ', which is then $q' - q$. If we express the Clebsch-Gordan coefficients in terms of Wigner coefficients with the aid of (5 2), then, apart from a trivial factor, the sum (27.1) will be reduced to the following jm-coefficient

$$C = \sum (-1)^{s_1 - m_1 + s_2 - m_2 + s_1' - m_1' + s_2' - m_2' + l_1' - \mu_1' + l_2' - \mu_2' + J_1 - M_1 + J_2 - M_2} \times$$

$$\times \begin{pmatrix} k & s_2 & s_1 \\ -q & -m_2 & m_1 \end{pmatrix} \begin{pmatrix} k' & s_2' & s_1' \\ q' & m_2' & -m_1' \end{pmatrix} \begin{pmatrix} l_1' & s_1' & J_1 \\ \mu_1' & m_1' & -M_1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} l_1 & s_1 & J_1 \\ -\mu_1 & -m_1 & M_1 \end{pmatrix} \begin{pmatrix} l_2 & s_2 & J_2 \\ \mu_2 & m_2 & -M_2 \end{pmatrix} \begin{pmatrix} l_2' & s_2' & J_2 \\ -\mu_2' & -m_2' & M_2 \end{pmatrix} \begin{pmatrix} l_1' & l_2' & l' \\ -\mu_1' & \mu_2' & -\mu' \end{pmatrix}. \tag{27.2}$$

The diagram of this jm-coefficient is given in Figure 27.1.

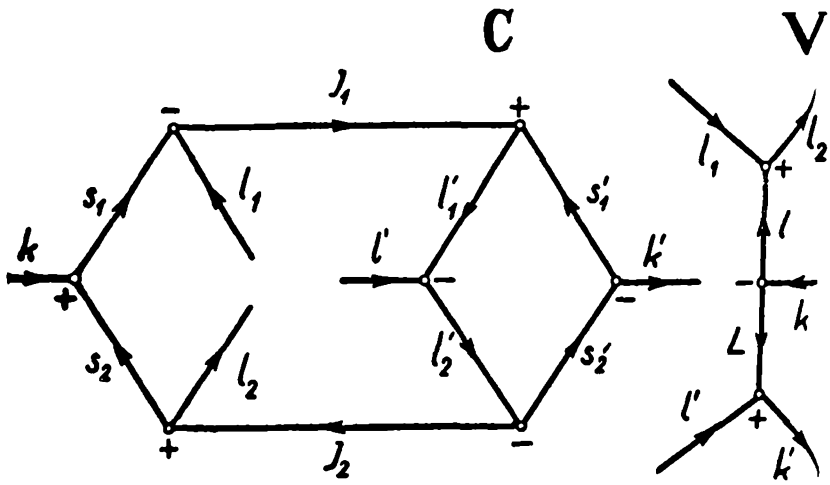

Figure 27.1

Let us now expand the jm-coefficient C in generalized Wigner coefficients with the help of (13.5), choosing for the generalized Wigner coefficient the expression V corresponding to the diagram shown in Figure 27.1. We have

$$C = \sum_{Ll} (L)(l) R V. \tag{27.3}$$

To obtain the diagram of the j-coefficient R, one must join the lines l_1, l_2, k, k', l' in diagrams C and V, with the directions as in diagram C. The diagram R thus obtained is shown in Figure 27.2. This diagram is separable on the three lines J_1, L, J_2. The j-coefficient R therefore decomposes into the product of two j-coefficients

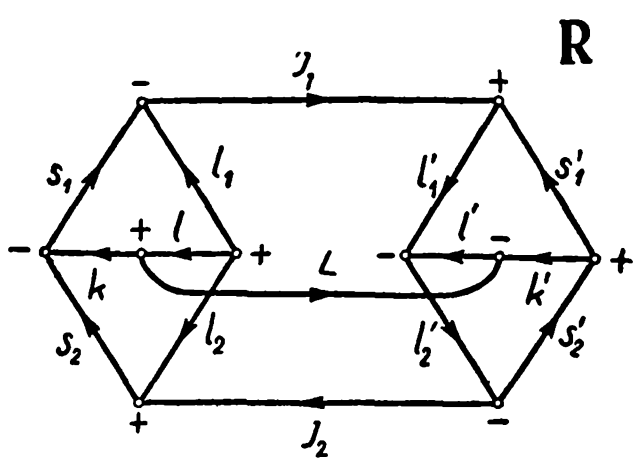

Figure 27.2

with fewer parameters. So as to make use of (14.5) we change the direction of the line J_2. We then obtain the new j-coefficient R', where

$$R=(-1)^{2J_2}R'. \tag{27.4}$$

From (14.5) we have

$$R'=X_1\cdot X_2. \tag{27.5}$$

The diagrams X_1 and X_2 (Figure 27.3) are not separable on fewer than four lines. They

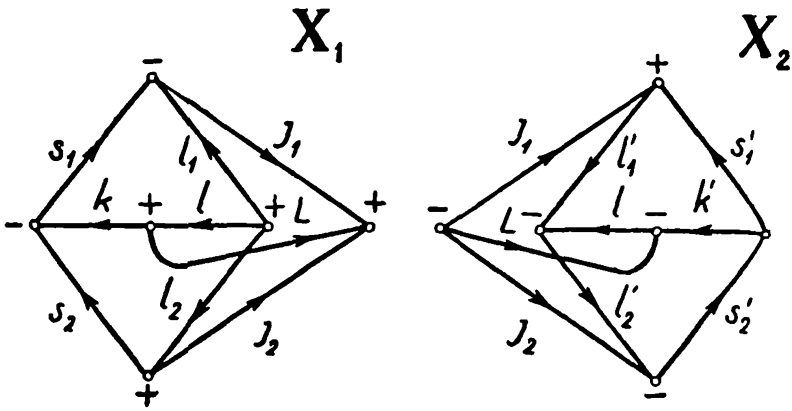

Figure 27.3

correspond to $9j$-coefficients. This will readily be seen by re-drawing, say, diagram X_2 in the form of a hexagon (Figure 27.4). Comparing this diagram with diagram 18.1, we find

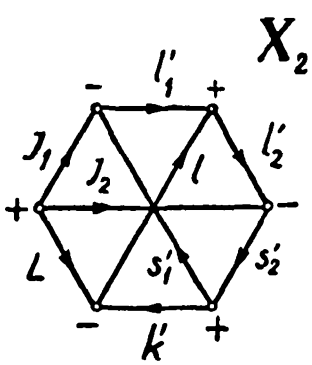

Figure 27.4

$$X_2=(-1)^{\varphi_2}\begin{Bmatrix} J_1 & l'_1 & s'_1 \\ L & l' & k' \\ J_2 & l'_2 & s'_2 \end{Bmatrix}, \tag{27.6}$$

where

$$\varphi_2=l'_1+l'_2+l+2J_1+2J_2+ \\ +2L+2s'_1+2k'+2l'_2. \tag{27.6a}$$

Analogously we obtain

$$X_1=(-1)^{\varphi_1}\begin{Bmatrix} J_1 & l_1 & s_1 \\ L & l & k \\ J_2 & l_2 & s_2 \end{Bmatrix}, \tag{27.7}$$

where

$$\varphi_1=2l_1+2l_2+2s_1+2l. \tag{27.7a}$$

The algebraic expression for the generalized Wigner coefficient V is

$$V=\sum_{\mu M}(-1)^{l-\mu+L-M}\begin{pmatrix} l_1 & l & l_2 \\ -\mu_1 & -\mu & \mu_2 \end{pmatrix}\begin{pmatrix} l & k & L \\ \mu & -q & M \end{pmatrix}\times\begin{pmatrix} l' & k' & L \\ -\mu' & q' & -M \end{pmatrix} \tag{27.8}$$

Here summation is actually limited to the single term in which $\mu=\mu_2-\mu_1$ and

$M = q' - \mu' = q + \mu_1 - \mu_2$. Introducing all the expressions obtained into (27.3), we obtain, for $\mu_1 = \mu_2 = 0$,

$$C\Big|_{\mu_1=\mu_2=0} = (-1)^{l_1+l_2+l'-q} \sum_{L} (-1)^{L+l} (L)(l) \begin{Bmatrix} J_1 & l_1 & s_1 \\ L & l & k \\ J_2 & l_2 & s_2 \end{Bmatrix} \begin{Bmatrix} J_1 & l_1' & s_1' \\ L & l' & k' \\ J_2 & l_2' & s_2' \end{Bmatrix} \times$$

$$\times \begin{pmatrix} l_1 & l & l_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & k & L \\ 0 & -q & q \end{pmatrix} \begin{pmatrix} l' & k' & L \\ -\mu' & q' & -q \end{pmatrix}. \qquad (27.9)$$

A comparison with (25.8) shows that in (27.9) we have, apart from a trivial factor, the so-called $8j$-coefficient (see the end of section 25).

28. Summation of a product of j-coefficients.

In practical calculations one frequently encounters sums of products of j-coefficient over one or several parameters. The result of this summation is again a j-coefficient, the diagram of which may be derived from the rules in section 15. In certain instances this diagram may be separable on fewer then four lines, the j-coefficient one obtains will then decompose into the product of simpler j-coefficients. To calculate the j-coefficients obtained upon summation, they may conveniently be expressed in terms of sums of products of j-coefficients with fewer parameters. Various expressions of this kind can be obtained by the method of separation elaborated in section 14. For numerical calculations it is preferable to have sums of the least possible multiplicity, containing j-coefficients with the smallest possible number of parameters. Such formulas are given for $9j$-, $12j$- and $18j$-coefficients in terms of $6j$- and $9j$-coefficients in sections 17-20 and in appendix 6. As extensive tables are available only for the $6j$-coefficients, the $9j$-coefficients entering into the expressions under consideration may be expressed in terms of the latter, although this will increase the multiplicity of the sum.

If the original sum is n-fold and the resulting diagram is not separable on less than $n+3$ lines, the multiplicity of the sum cannot be reduced. However, by determining and then separating the diagram which represents such a sum, it may be possible to express the sum in different forms and sometimes even to simplify it. We shall illustrate this with a simple example.

It is easy to verify the following identity /Vizbaraite and Jucys 1959/ for the $12j$-coefficient of the second kind

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = (-1)^{j_1-j_2-j_3+j_4} \sum_{x} (x) \begin{Bmatrix} l_4 & l_1 & x \\ k_3 & j_1 & l_2 \\ j_3 & k_1 & l_3 \end{Bmatrix} \begin{Bmatrix} l_4 & l_1 & x \\ k_4 & j_3 & l_2 \\ j_4 & k_2 & l_3 \end{Bmatrix}. \qquad (28.1)$$

The graphical representation of this $12j$-coefficient is given in Figure 19.2. Assume that we are trying to evaluate the sum in the right-hand side of (28.1) and have found its diagram. It is separable only on four $(1+3)$ lines. It would therefore seem that the sum cannot be simplified. If, however, after separation on the lines l_1, l_2, l_3

and l_4 one joins the line l_1 to l_2 and l_3 to l_4, then the two resulting parts will be separable on three lines. This reduces to a single sum of a product of four $6j$-coefficients (formula 19.3)). The multiplicity of the sum has not changed, but the summed product has been simplified instead of the product of two $9j$-coefficients (each of which, from (18.2), is a summed product of three $6j$-coefficients), we have a product of $6j$-coefficients.

In most cases summation results in a marked simplification of the expression under consideration. As an example of this, consider the following formula /Arima et al. 1954/

$$\sum_{xy}(x)(y)(-1)^{y-j_2+j_1-j}\begin{Bmatrix} k_2 & x & k' \\ j_2 & l_2 & j_2' \end{Bmatrix}\begin{Bmatrix} k_2 & x & k' \\ y & k_1 & k \end{Bmatrix}\begin{Bmatrix} k_1 & y & k' \\ j_1 & l_1 & j_1' \end{Bmatrix}\begin{Bmatrix} j_2 & j_2' & x \\ j_1 & j_1' & y \\ j & j' & k \end{Bmatrix} =$$

$$= (-1)^{k+k_1-j_1-l_2}\begin{Bmatrix} j' & j_1' & j_2' \\ j & l_1 & l_2 \\ k & k_1 & k_2 \end{Bmatrix}\begin{Bmatrix} l_1 & l_2 & j \\ j_2 & j_1 & k' \end{Bmatrix}, \qquad (28.2)$$

which expresses a double sum as a simple product of $6j$- and $9j$-coefficients. We shall illustrate the graphical summation of j-coefficients with this characteristic example.

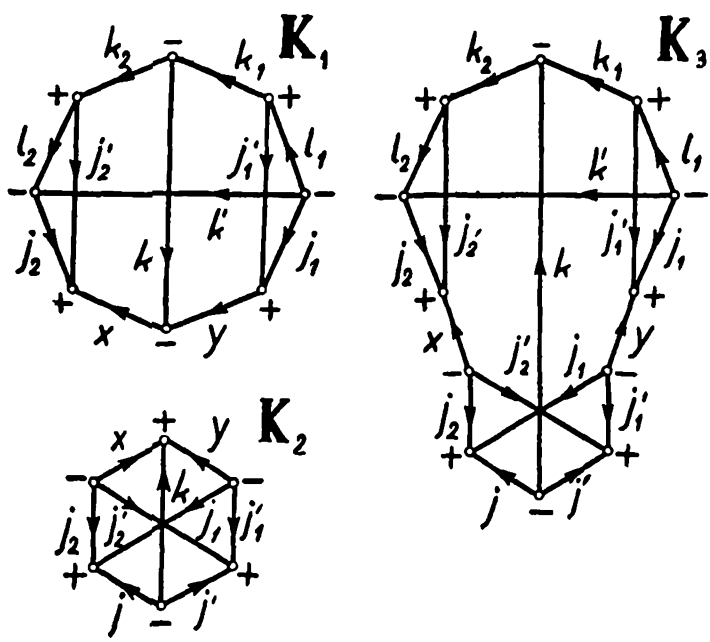

Figure 28.1

Let us denote the required sum by K and note that from the "multiplication rule" of section 15 the product of the three $6j$-coefficients in (28.2) may be represented by the diagram K_1 in Figure 28.1. K can then be written as

$$K = \sum_{xy}(-1)^{y-k_2+j_1-j}(x)(y)K_1K_2, \qquad (28.3)$$

where K_2 is the $9j$-coefficient in Figure 28.1. We change the directions of the lines K_1 and k in diagram y, and obtain

$$K_1=(-1)^{2k+2y}K_1'=(-1)^{2k+2j_1+2j_1'}K_1', \tag{28.4}$$

where K_1' is the j-coefficient represented by the diagram K_1 after the directions of these two lines have been changed. Using the multiplication rule for the product $K_1'\cdot K_2$, we obtain the diagram $K_3=K_1'K_2$. The original sum then becomes

$$K=\sum_{xy}(x)(y)(-1)^{y-j_1+j_1'-j-2k+2j_1+2j_1'}K_3. \tag{28.5}$$

We change the sign of one of the nodes of the line y so as to be able to use the "summation rule" of section 15. This gives

$$K=(-1)^{-j_1-j_1-j-2k+2j_1+2j_1'}\sum_{xy}(x)(y)K_3'. \tag{28.6}$$

The summation is carried out by dropping the lines x and y from diagram K_3' together with the adjacent nodes and joining corresponding lines, as a result we obtain

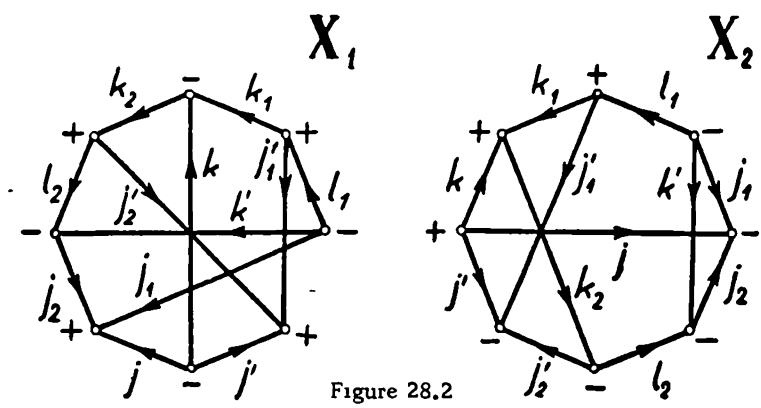

Figure 28.2

the diagram X_1 in Figure 28.2. Re-drawing it as X_2 and changing the direction of the line l_1, we find

$$X_1=X_2=(-1)^{2l_1}X_2'. \tag{28.7}$$

Separating X_2' on the lines l_1, l_2 and j_3 we obtain the two diagrams Y_1 and Y_2 in

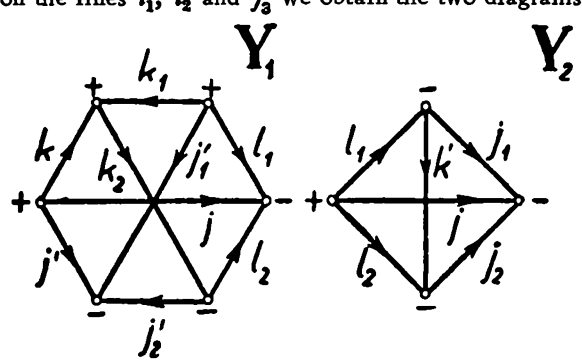

Figure 28.3

Figure 28.3. A comparison of 18.1 and 16.1 gives

$$Y_1 = (-1)^{k+k_1+l_2+j_3} \begin{Bmatrix} j' & j_1' & j_2' \\ j & l_1 & l_2 \\ k & k_1 & k_2 \end{Bmatrix} \tag{28.8}$$

and

$$Y_2 = (-1)^{j-j_1+j_2} \begin{Bmatrix} l_1 & l_2 & j \\ j_2 & j_1 & k' \end{Bmatrix}. \tag{28.9}$$

Finally, introducing

$$\sum_{xy} (x)(y) K_3' = X_2 = (-1)^{2l_1} Y_1 Y_2, \tag{28.10}$$

in (28.6), we obtain the sum (28.2).

We end this section with an example of the simplification of a jm-coefficient when one of the parameters is zero. Assume that $k_2 = 0$ in the $12j$-coefficient of the second kind (Figure 19.2). Let us indicate the directions of the lines and the signs of the nodes adjoining the line k_2 to correspond with the diagram in Figure 14.5. To do this we must change the directions of l_3 and l_1 and the sign of the node at which the lines $l_1 j_3 k_2$ meet. For $k_2 = 0$ we obtain for the $12j$-coefficient of the second kind

$$(-1)^{2j_3+2l_1+l_1+j_3} \left[(l_1)(l_3)\right]^{-\frac{1}{2}} \delta(j_4, l_3)\, \delta(j_3, l_1) \cdot X. \tag{28.11}$$

Separating the diagram X (Figure 28.4) on the lines k_1, k_3 and k_4 (for this

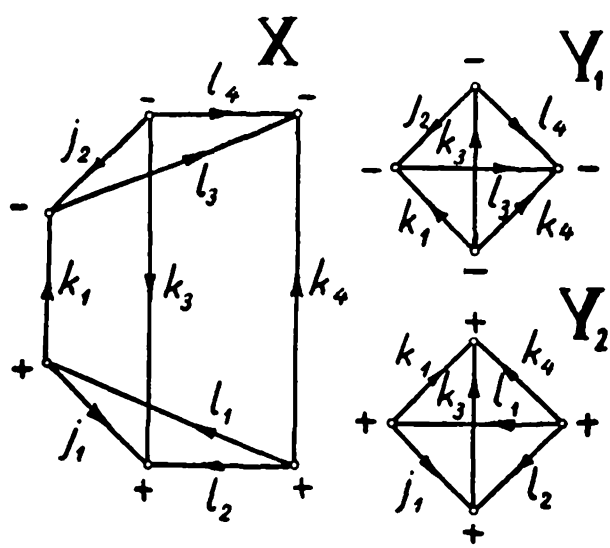

Figure 28.4

we must first change the direction of k_3), we find

$$X = (-1)^{2k_3} Y_1 Y_2. \tag{28.12}$$

Comparing the diagrams Y_1 and Y_2 with the diagrams in Figure 16.1, we find

$$Y_1 = (-1)^{j_2-k_1-k_3-l_2} \begin{Bmatrix} j_2 & k_1 & l_3 \\ k_4 & l_4 & k_3 \end{Bmatrix}, \tag{28.13}$$

$$Y_2 = (-1)^{j_1+k_1-k_3+l_2} \begin{Bmatrix} k_1 & j_1 & l_1 \\ l_2 & k_4 & k_3 \end{Bmatrix}. \tag{28.14}$$

From (28.11)-(28.14) we finally obtain

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & 0 & k_3 & k_4 \end{bmatrix} = (-1)^{j_1-j_2+l_2-l_1} \left[(l_1)(l_3)\right]^{-\frac{1}{2}} \begin{Bmatrix} j_2 & k_1 & l_3 \\ k_4 & l_4 & k_3 \end{Bmatrix} \begin{Bmatrix} k_1 & j_1 & l_1 \\ l_2 & k_4 & k_3 \end{Bmatrix}, \tag{28.15}$$

which is essentially identical with (19.10)

29. Summation of a product of Wigner coefficients and j-coefficients

In this section we will give an example of the graphical summation of a product in which both Wigner coefficients and j-coefficients appear explicitly. The summation may extend over the angular momentum and the magnetic quantum numbers. Consider the sum

$$H = \sum_{nrkq} (-1)^{n+r+k-q} (n)(r)(k) \begin{pmatrix} l_1 & l_1' & k \\ n_1 & n_1' & q \end{pmatrix} \begin{pmatrix} l_2 & l_2' & k \\ -n_2 & -n_2' & -q \end{pmatrix} \times$$

$$\times \begin{Bmatrix} j_1 & j_2 & J \\ j_2' & j_1' & r \end{Bmatrix} \begin{Bmatrix} n & k & r \\ s & l_1 & j_1 \\ s' & l_1' & j_1' \end{Bmatrix} \begin{Bmatrix} n & k & r \\ s' & l_2 & j_2 \\ s & l_2' & j_2' \end{Bmatrix}. \tag{29.1}$$

Summation over the magnetic quantum number q gives the jm-coefficient V (the generalized Wigner coefficient of Figure 29.1). The required sum may then be written as

$$H = \sum_{nrk} (-1)^{n+r} (n)(r)(k) V W X_1 X_2, \tag{29.2}$$

The individual factors in the above are represented in the diagrams in Figure 29.1, where for convenience identical nodes have been indicated by (the same) numbers.

Let us first sum over k, applying the rule for the summation of a product to $V X_1 X_2$. In order to do this, we will have to change the directions of the lines n, k, r, l_2 and l_2' in the diagram X_2. This will bring us to the diagram X_2'. Now, $X_2 = (-1)^{2l_2+2l_2'} X_2'$, since $2n+2k+2l$ is an even number. We have

$$H = \sum_{nr} (-1)^{n+r} (n)(r) \cdot (-1)^{2l_2+2l_2'} W \cdot \sum_{k} (k) V X_1 X_2' =$$

$$= (-1)^{2l_2+2l_2'} \sum_{nr} (-1)^{n+r} (n)(r) W A. \tag{29.3}$$

The diagram of the jm-coefficient A is obtained from the rule for the summation of a product. Separating the lines adjoining the line k in diagrams V, X_1 and X_2 and

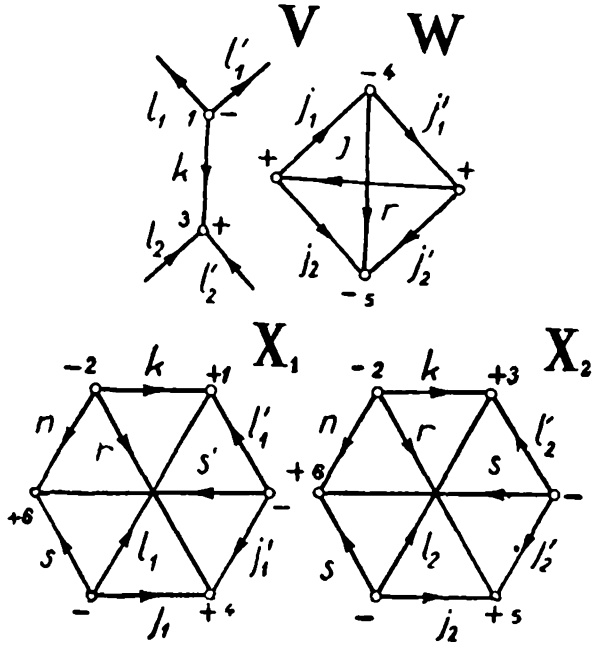

Figure 29.1

dropping all parts which contain the lines k, we obtain the diagrams in Figure 29.2. The numbers at the ends of the lines indicate the number of the node to which the line

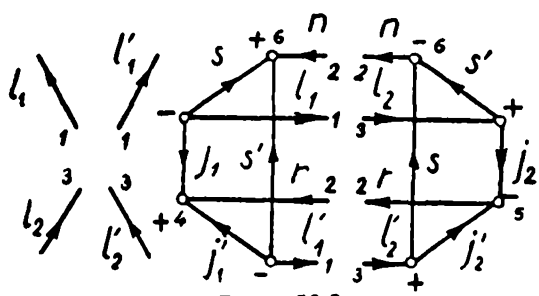

Figure 29.2

converged before separation. Joining the free ends of corresponding lines*, we obtain the diagram in Figure 29.3. The jm-coefficient A may be summed over n with the "summation rule" for one jm-coefficient (section 15). To do this one must first change the orientation of one of the nodes 6 so as to make them the same, and change

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* In certain cases, if the lines are labelled by angular momenta alone, there may be more than one pair of free lines with the same index in the separated diagrams. An additional label would then be required to define "corresponding lines" uniquely, and this is the number of the node to which the line converged before separation of the diagrams. It would be seen that lines which correspond "non-significantly" converged to different nodes before separation.

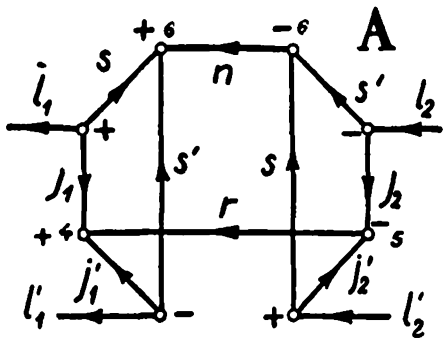

Figure 29.3

the directions of the lines s and s' in the right side of the diagram. This gives the new diagram A', where $A = (-1)^{2s+2s'+n+s+s'} A' = (-1)^{s+s'-n} A'$. We therefore have

$$H = (-1)^{2l_1+2l_2} \sum_r (-1)^{s+s'+r} (r) W \sum_n (n) A' =$$

$$= (-1)^{2l_1+2l_2'+s+s'} \sum (r) (-1)^r W \cdot B. \qquad (29.4)$$

The diagram of the jm-coefficient B is given in Figure 29.4. Summation over r is carried out by applying the rule for the summation of a product to $W \cdot B$. One should

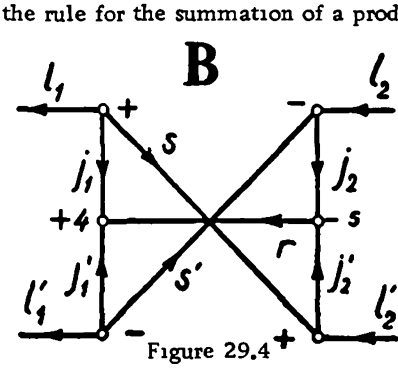

Figure 29.4

first change the orientation of the node 5 and the direction of the line W in the diagram j_1, and the directions of the lines j_2 and j_2' in the diagram B. This gives the diagrams W' and B', so that

$$H = (-1)^{2l_1+2l_2'+s+s'+2j_1+j_2+j_2'} \times \sum_r (r) W' \cdot B'. \qquad (29.5)$$

Dropping the parts containing the line r in the diagrams W' and B', we obtain the diagrams in Figure 29.5. Joining the ends of corresponding lines (converging to the same nodes), we obtain the diagram C (Figure 29.6). Thus we have, finally,

$$H = (-1)^{j_2-j_2'-s-s'+2J} C. \qquad (29.6)$$

Now, if we wish we may write the jm-coefficient C as a sum of products of six Wigner coefficients. Then the conditions that they do not vanish can be used to

reduce the number of summation parameters to two, instead of the seven in the original

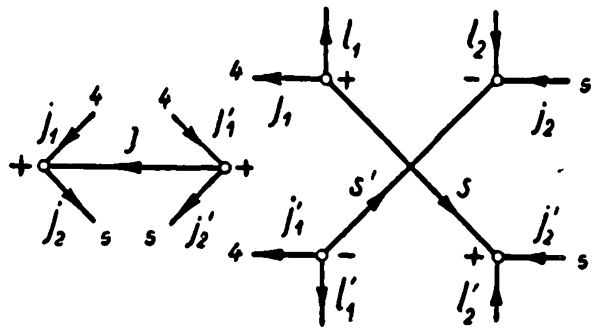

Figure 29.5

We will next show how the three parameters n, k and r can be summed over simultaneously. We note that in the process of summation we had separated all lines adjacent to the lines n, k and r, dropped all parts containing these lines and joined the ends of corresponding lines converging to the same nodes in the remaining parts. We can therefore proceed as follows. We orient the nodes and direct the lines adjacent to the lines of the summation parameters in such a way that the summation may be carried out over any of the parameters. In our example, this means that we must change the orientation of the node 5 and the direction of the line j_1 in the diagram W, and the orientation of the node 6 and the directions of the lines n, k, r, s', j_2, l_2, l'_2, s, j''_2 in the diagram X_2 (i.e., immediately introduce the changes which were earlier carried out through various intermediate diagrams). Next we must separate the lines adjacent to the lines of the summation parameters and drop

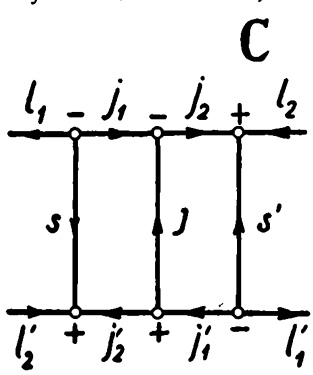

Figure 29.6

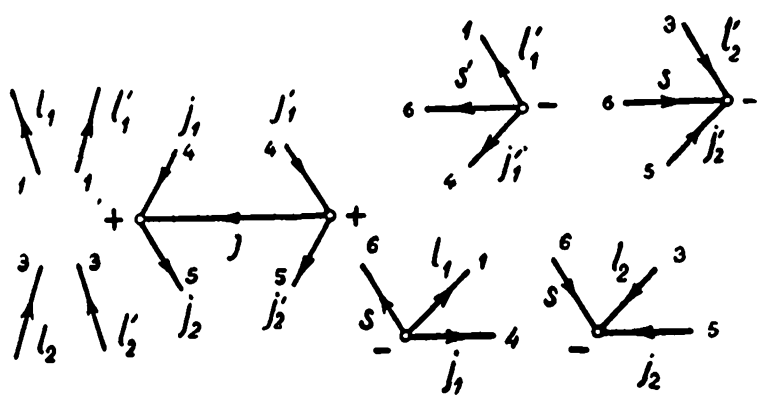

Figure 29.7

those parts of the diagrams which contain these lines. In this example we are then left with the diagrams in Figure 29.7. Joining the ends of corresponding lines converging to the same nodes, we obtain the diagram of the sum. For our example, this immediately gives the diagram C. In the general case, the validity of this rule for obtaining the diagram of a sum is evident if one can assume that when the summation is performed step by step, after the summation over each parameter it is possible to further sum over one of the remaining parameters with the aid of our rules. For this to hold true the original expression to be summed must satisfy certain conditions, which, however, have not yet been established. All sums encountered in practice satisfy this condition. Presumably it is related to the requirement that the result of the summation be a jm-coefficient.

30. Choice of a method of calculation

In sections 26-29, it was shown how various transformations of sums of j-coefficients and Wigner coefficients are carried out by the graphical method. These transformations may also be carried out algebraically. This requires the use of identities such as the ones given in appendices 6 and 7, which are usually obtained from matrix identities as described in sections 24 and 25.

Each method possesses certain advantages and disadvantages.

The algebraic method is convenient only when one has a sufficiently complete list of sum rules and transformation formulas for j- and jm-coefficients. The main drawback of this method is its lack of generality for each concrete case one must choose the appropriate transformations leading to the required result, often a very difficult task. It should be noted that once the method of transformation is known, the use of algebraic formulas gives the final result rather rapidly.

When the graphical method is used, j and m are summed over with the help of two standard rules which are applicable in all cases without exception. The number of elementary operations is somewhat larger than when algebraic formulas are used. The result of the summation is a jm-coefficient obtained in the form of a diagram. The method of transformation is easy to guess owing to the clarity of the graphical operations.

In certain cases it is convenient to combine the two methods. Use of non-oriented diagrams makes it possible to establish the result very rapidly up to the phase factor. If one can thus discern the necessary algebraic transformations, the phase factor can be found by carrying them out. Usually, however, this is possible only in cases which are not very complicated. Using non-oriented diagrams, one can foresee the usefulness of algebraic transformations, while it is usually impossible to tell by inspection whether or not an algebraic sum can be evaluated.

As an illustration of the use of both methods in conjunction, let us consider the following example

$$P=\sum_{x}(x)\begin{Bmatrix} j_3 & j & l_1 \\ l_2 & x & k_3 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & x \\ k_1 & k_2 & k_3 \end{Bmatrix}. \tag{30.1}$$

We draw the $6j$-coefficient and the $9j$-coefficient with the lines of the summed parameters lying close to each other (Figure 30.1), but without paying attention to

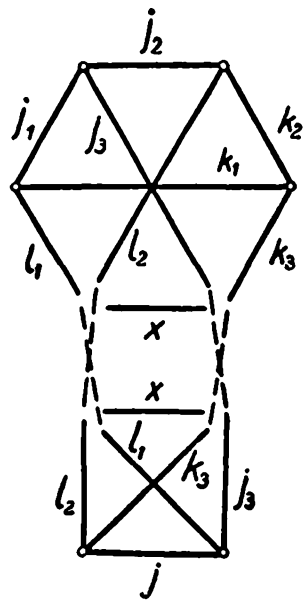

Figure 30.1

the directions of the lines or to the signs of the nodes. We drop the parameter x and the nodes adjacent to it, and join corresponding lines, as indicated in Figure 30.1 by the the dotted line. We re-draw this diagram as in Figure 30.2. This last diagram is

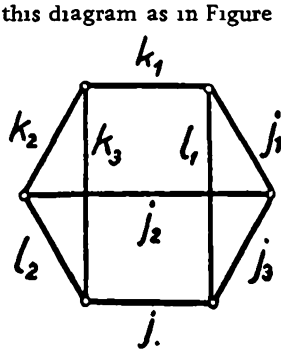

Figure 30.2

separable on the three lines j_2, k_1 and j. Further, again disregarding the directions of lines and signs of nodes, we separate it into two parts and obtain two $6j$-coefficients (Figure 30.3). We then write

$$P=(-1)^{\varphi}\begin{Bmatrix} j_1 & l_1 & k_1 \\ j & j_2 & j_3 \end{Bmatrix}\begin{Bmatrix} j_2 & j & k_1 \\ k_3 & k_2 & l_2 \end{Bmatrix}, \tag{30.2}$$

where φ is to be determined algebraically.

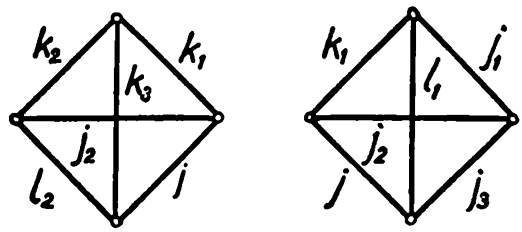

Figure 30.3

From (18.5) and (18.2) we may write

$$P=\sum_x (x)\begin{Bmatrix} j_3 j & l_1 \\ l_2 & x & k_3 \end{Bmatrix}\begin{Bmatrix} j_2 & j_3 & j_1 \\ l_2 & x & l_1 \\ k_2 & k_3 & k_1 \end{Bmatrix}=\sum_{xy}(x)(y)(-1)^{2y}\begin{Bmatrix} j_3 j & l_1 \\ l_2 & x & k_3 \end{Bmatrix}\begin{Bmatrix} j_2 j_3 & j_1 \\ l_1 & k_1 & y \end{Bmatrix}\times$$

$$\times\begin{Bmatrix} l_2 & x & l_1 \\ j_3 & y & k_3 \end{Bmatrix}\begin{Bmatrix} k_2 & k_3 & k_1 \\ y & j_2 & l_2 \end{Bmatrix}. \tag{30.3}$$

Here in accordance with the orthonormality condition (24.2) of the $6j$-coefficient we have, for the first and the third $6j$-coefficient,

$$(y)\sum_x (x)\begin{Bmatrix} j_3 j & l_1 \\ l_2 & x & k_3 \end{Bmatrix}\begin{Bmatrix} l_2 & x & l_1 \\ j_3 & y & k_3 \end{Bmatrix}=\delta(y,j). \tag{30.4}$$

Introducing this equality in (30.3) and comparing with (30.2), we find

$$\varphi=2j. \tag{30.5}$$

Then (30.1), (30.2) and (30.5) finally give

$$\sum_x (x)\begin{Bmatrix} j_3 j & l_1 \\ l_2 & x & k_3 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & x \\ k_1 & k_2 & k_3 \end{Bmatrix}=(-1)^{2j}\begin{Bmatrix} j_1 j_2 j_3 \\ j & l_1 & k_1 \end{Bmatrix}\begin{Bmatrix} k_1 & k_2 & k_3 \\ l_2 & j & j_2 \end{Bmatrix}. \tag{30.6}$$

The following formula may be established in the same way

$$\sum_x (-1)^{j_1+j_2+j_3+k_1+k_2+k_3+l_1+l_2+x}\begin{Bmatrix} j_3 j & l_2 \\ l_1 & x & k_3 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & x \\ k_1 & k_2 & k_3 \end{Bmatrix}=$$

$$=(-1)^{2j}\begin{Bmatrix} j_1 j_2 j_3 \\ l_2 j & k_2 \end{Bmatrix}\begin{Bmatrix} k_1 & k_2 & k_3 \\ j & l_1 & j_1 \end{Bmatrix}. \tag{30.7}$$

This formula can be obtained from (30.6) by permuting the first two columns in the $9j$-coefficient and subsequently renumbering the parameters in these columns

We must point out that these examples are rather simple, and neither method of calculation will present particular difficulties. These examples were merely used to show how to proceed when calculations must be carried out. Obviously, it may sometimes be useful to carry out the calculations by both methods entirely for purposes of checking.

Chapter VII

IRREDUCIBLE TENSOR OPERATORS AND EXPRESSIONS FOR THEIR MATRIX ELEMENTS

One of the principal applications of the mathematical apparatus discussed in the preceding chapters is the calculation of matrix elements of operators. This calculation may be simplified by the use of irreducible tensor operators. The corresponding methods have been developed by Wigner /1931/ and Racah /1942/, and it is in fact in their application that the mathematical apparatus described earlier was evolved. Use of these methods substantially simplifies the calculation of matrix elements.

The present chapter briefly discusses methods of calculating the matrix elements of irreducible tensor operators. In section 31 we consider general aspects of this method. The following section is devoted to the products of irreducible tensor operators. Section 33 gives expressions for the matrix elements of tensor products of two irreducible tensor operators. The next section gives an example of calculation of matrix elements of the tensor product of four irreducible tensor operators. The last section deals with irreducible double tensor operators and their products.

31. Irreducible tensor operators and their properties

The question of decomposing tensor operators to an irreducible set is closely related to the reduction of the tensor representation of the three-dimensional rotation group. The 3^r components of the tensor $T_{i_1 i_2 \ldots i_r}$ of rank r form the basis of the tensor representation $\mathcal{T}$, which is the direct product of the vector representations $\mathcal{V}$ of the rotation group

$$\mathcal{T} = \mathcal{V} \times \mathcal{V} \times \ldots \times \mathcal{V} \quad (r \text{ times}) \tag{31.1}$$

Decomposing the representation $\mathcal{T}$, we obtain

$$\mathcal{T} = \sum_k a_k \mathcal{D}_k \tag{31.2}$$

Here a_k indicates the number of times the irreducible representation $\mathcal{T}$ appears in the decomposition of $\mathcal{D}_k$. We have the obvious equality

$$\sum_k a_k (2k+1) = 3^r . \tag{31.3}$$

The decomposition of $\mathcal{T}$ amounts to transformation to a new basis consisting of the quantities T^k_q $(q = -k, \ldots, k)$ which are linear combinations of the quantities $T_{i_1 \ldots i_r}$. The set of $(2k+1)$ quantities T^k_q with fixed k forms the basis of the irreducible representation $\mathcal{D}_k$ of the rotation group. This set is the irreducible tensor T^k

of rank k with components T^k_q. The transformation properties of an irreducible tensor can be expressed by its commutators with the infinitesimal rotation operators. From (1.4) this reduces to the following commutator with an angular momentum

$$\left[J_z, T^k_q\right] = qT^k_q, \tag{31.4a}$$

$$\left[J_x \pm iJ_y, T^k_q\right] = \left[(k \mp q)(k \pm q + 1)\right]^{\frac{1}{2}} T^k_{q\pm 1}. \tag{31.4b}$$

These commutators are naturally identical with the commutators of the spherical functions Y_{kq}, since the transformation properties of these quantities are the same. This considerably simplifies the calculation of matrix elements of irreducible tensors as compared to those of reducible ones.

In the main, contraction, symmetrization and anti-symmetrization are the processes used to decompose a reducible tensor into a set of irreducible ones. The details of this reduction may be found in manuals on the theory of group representations (e.g., Gel'fand et al. /1958/). We will give a summary of the results for the cases of greatest practical importance, namely $r = 0, 1, 2$. For $r = 0$ we have the scalar

$$T = T^0_0. \tag{31.5}$$

For $r = 1$ (vector) the tensor representation $\mathcal{T} = \mathcal{V}$ and is therefore equivalent to $\mathcal{D}_1$. We have

$$T^1_0 = T_z$$

$$T^1_{\pm 1} = \mp \frac{1}{\sqrt{2}} (T_x \pm iT_y). \tag{31.6}$$

For $r = 2$ (second rank tensor) $\mathcal{T} = \mathcal{V} \times \mathcal{V}$ and is therefore equivalent to $\mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2$. A reducible tensor of the second rank can thus be decomposed into irreducible tensors of ranks $k = 0, 1, 2$. Their components are given by

$$T^0_0 = -\frac{1}{\sqrt{3}} (T_{xx} + T_{yy} + T_{zz}) \tag{31.7a}$$

$$T^1_0 = \frac{i}{\sqrt{2}} (T_{xy} - T_{yx}) \tag{31.7b}$$

$$T^1_{\pm 1} = \mp \frac{i}{2} \left[(T_{yz} - T_{zy}) \mp i(T_{zx} - T_{xz})\right]$$

$$T^2_0 = \frac{1}{\sqrt{6}} \left[2T_{zz} - T_{xx} - T_{yy}\right]$$

$$T^2_{\pm 1} = \frac{1}{2} \left[\mp (T_{xz} + T_{zx}) - i(T_{yz} + T_{zy})\right] \tag{31.7c}$$

$$T^2_{\pm 2} = \frac{1}{2} \left[T_{xx} - T_{yy} \pm i(T_{xy} + T_{yx})\right].$$

The normalization in (31.6) and (31.7) is so chosen that

$$\sum_{i_1 \cdots i_r} T^{\dagger}_{i_1 i_2 \cdots i_r} T_{i_1 i_2 \cdots i_r} = \sum_{kq} (-1)^q T^k_{-q} T^{\dagger}_q. \tag{31.8}$$

Let us consider the matrix of the irreducible tensor T^k in the system of eigenfunctions of angular momentum

$$(\alpha jm|\, T^k_q \,|\alpha' j' m')$$

Here α is the set of quantum numbers which completes the set. The dependence of this matrix element on m, m' and q is fully determined by the transformation (geometrical) properties of the tensor T^k, whence it follows /Wigner 1931, Racah 1942/ that (the Wigner-Eckart theorem)

$$(\alpha jm|\, T^k_q \,|\alpha' j' m') = (-1)^{j-m} \begin{pmatrix} j & k & j' \\ -m & q & m' \end{pmatrix} (\alpha j \| T^k \| \alpha' j'). \qquad (31.9)$$

The above is the fundamental formula in the algebra of tensor operators. The phase has been so chosen that when the graphical methods of Chapter III are used one automatically obtains the phase factors required there. The quantity $(\ \|\ \|\ )$ is independent of m, m' and q and is called the reduced matrix element of the tensor T^k. The set of these quantities forms the reduced matrix of the tensor. For the reduced matrix of the hermitian conjugate tensor one readily finds /Racah 1942/

$$(\alpha j \| T^{k^+} \| \alpha' j') = (-1)^{j-j'} (\alpha' j' \| T^k \| \alpha j)^*. \qquad (31.10)$$

If the tensor is hermitian /Racah 1942/, then

$$T^{k^+}_q = (-1)^q T^k_{-q} \qquad (31.11)$$

and (31.10) reduces to

$$(\alpha j \| T^k \| \alpha' j') = (-1)^{j'-j'} (\alpha' j' \| T^k \| \alpha j), \qquad (31.12)$$

i. e., the reduced matrix is not hermitian. It is useful to bear in mind that for the scalar operator T^0

$$(\alpha jm|\, T^0 \,|\alpha' j' m') = \delta(j,j')\,\delta(m,m')\,(j)^{-\frac{1}{2}} (\alpha j \| T^0 \| \alpha' j). \qquad (31.9a)$$

If the scalar operator is $T^0_0 = 1$, then its diagonal element is equal to unity and from (31.9a) we find

$$(\alpha j \| 1 \| \alpha' j) = \delta(\alpha, \alpha')\,(j)^{\frac{1}{2}}. \qquad (31.13)$$

A similar method is generally used for the calculation of reduced matrix elements. Using the definition of the operator, one calculates the simplest matrix element directly, the reduced matrix element can then be found from (31.9).

32. Tensor products

Let us consider the two irreducible tensors T^{k_1} and U^{k_2}. Multiplying all the $2k_1 + 1$ components of the first tensor by all the $2k_2 + 1$ components of the second, we obtain the $(2k_1 + 1)\cdot(2k_2 + 1)$ quantities $[T^{k_1} \times U^{k_2}]_{q_1 q_2}$. These quantities are components of the tensor $[T^{k_1} \times U^{k_2}]$, which is the direct product of the tensors T^{k_1} and U^{k_2}. Its components transform according to $\mathcal{D}_{k_1} \times \mathcal{D}_{k_2}$ and this tensor

therefore decomposes into irreducible tensors of ranks $k = |k_1 - k_2| \ldots k_1 + k_2$, which are called tensor products of rank k. In complete analogy with the coupling of angular momenta, the components of this product are given by the following formula /Racah 1951, Fano 1951/

$$\left[\mathbf{T}^{k_1} \times \mathbf{U}^{k_2}\right]^k_q = \sum_{q_1 q_2} T^{k_1}_{q_1} U^{k_2}_{q_2} (k_1 k_2 q_1 q_2 \mid k_1 k_2 k q). \tag{32.1}$$

It is easily verified that this tensor satisfies the conditions (31.4).

Using the unitarity property of the matrix of the Clebsch-Gordan coefficients, we obtain

$$T^{k_1}_{q_1} U^{k_2}_{q_2} = \sum_{kq} \left[\mathbf{T}^{k_1} \times \mathbf{U}^{k_2}\right]^k_q (k_1 k_2 k q \mid k_1 k_2 q_1 q_2). \tag{32.2}$$

The rank of the product can be zero only if $k_1 = k_2$. Then (32.1) becomes

$$\left[\mathbf{T}^k \times \mathbf{U}^k\right]^0_0 = \sum_q (k k q - q \mid k k 0 0)\, T^k_q\, U^k_{-q}. \tag{32.3}$$

Using (4.9b), we obtain further

$$\left[\mathbf{T}^k \times \mathbf{U}^k\right]^0_0 = (-1)^k (k)^{-\frac{1}{2}} \sum_q (-1)^q T^k_q\, U^k_{-q}. \tag{32.4}$$

This quantity is a scalar, and while it would have been natural to regard it as the definition of the scalar product of two tensors of the same rank, the traditional definition of a scalar product is not identical with (32.4), and is related to it in the following way

$$(\mathbf{T}^k \cdot \mathbf{U}^k) = \sum_q (-1)^q T^k_q\, U^k_{-q} = (-1)^k (k)^{\frac{1}{2}} \left[\mathbf{T}^k \times \mathbf{U}^k\right]^0_0. \tag{32.5}$$

The advantage of the above definition over (32.4) is that for the case of vectors ($k = 1$) it coincides with the usual definition of the scalar product of two vectors, this may readily be seen by introducing (31.6) in (32.5).

For $k_1 = k_2 = 1$ and $k = 1$ the definition (32.1) is not identical with the usual vector product of two vectors. The following relation may easily be verified

$$\left[\mathbf{A},\mathbf{B}\right] = -i\sqrt{2}\left[\mathbf{A}^1 \times \mathbf{B}^1\right]^1. \tag{32.6}$$

Here $\mathbf{A}$ and $\mathbf{B}$ are vectors and by $\mathbf{A}^1$ and $\mathbf{B}^1$ are meant the same vectors in the spherical basis (31.6). The left-hand side of (32.6) is the vector product of the given vectors.

Making use of the symmetry of the Clebsch-Gordan coefficients, one can readily verify the following relation

$$\left[\mathbf{T}^{k_1} \times \mathbf{U}^{k_2}\right]^{\dagger k}_q = (-1)^{k_1 + k_2 + k} (-1)^q \left[\mathbf{T}^{k_1} \times \mathbf{U}^{k_2}\right]^k_{-q} \tag{32.7}$$

provided $\mathbf{T}^{k_1}$ and $\mathbf{U}^{k_2}$ commute and are hermitian, i.e., satisfy (31.11). The above equality shows that the tensor product of two hermitian tensors is hermitian when and only when $k_1 + k_2 + k$ is an even integer, otherwise it is an anti-hermitian tensor. One such tensor is, for example, the product $[\mathbf{A}^1 \times \mathbf{B}^1]^1$.

Generalizing (32.1) to the case of an arbitrary number of tensors, we write

$$\left[T^{k_1} \times U^{k_2} \times \ldots Y^{k_n} \right]^k_q = \sum_{q_1 \ldots q_n} T^{k_1}_{q_1} U^{k_2}_{q_2} \ldots Y^{k_n}_{q_n} \times$$

$$\times \left(k_1 q_1 \ldots k_n q_n | (k_1 \ldots k_n)^A Kkq \right). \qquad (32.8)$$

Here instead of the usual Clebsch-Gordan coefficient we have the corresponding generalized coefficient defined in (6.16). K is the set of ranks of intermediate products, the number of which is n-2. Generalization of (32.2) brings us to the following formula

$$T^{k_1}_{q_1} U^{k_2}_{q_2} \ldots Y^{k_n}_{q_n} = \sum_{Kkq} \left[T^{k_1} \times U^{k_2} \times \ldots \times Y^{k_n} \right]^k_q \times$$

$$\times \left((k_1 \ldots k_n)^A Kkq | k_1 q_1 \ldots k_n q_n \right). \qquad (32.9)$$

The concept of a scheme of addition of angular momenta introduced in section 6 must be replaced by the concept of a scheme of multiplication of tensors (or of addition of ranks of component tensors) in the case of multiplication of tensors. The passage from one scheme of multiplication to another is carried out exactly as for the eigenfunctions of coupled angular momenta in accordance with (8.1), i.e., with the aid of the transformation matrix. As we know, the elements of the latter may be expressed in terms of the j-coefficients.

The algebra described above is valid independently of what coordinates are acted upon by the individual tensor operators in the product. When calculating matrix elements it is convenient to express products of tensor operators in a form in which the operators acting on the same coordinates are adjacent to each other. This may be accomplished by changing the scheme of multiplication of the tensors with the aid of a transformation matrix.

33. Expressions for matrix elements of products of tensor operators

Let us consider the matrix element of the product (32.1), in which the two operators act on the same coordinates. We have

$$\left(\alpha jm \left| \left[T^{k_1} \times U^{k_2} \right]^k_q \right| \alpha' j' m' \right) = \sum_{\alpha'' j'' m''} (k_1 k_2 q_1 q_2 | k_1 k_2 kq) (\alpha jm | T^{k_1}_{q_1} | \alpha'' j'' m'') \times$$

$$\times (\alpha'' j'' m'' | U^{k_2}_{q_2} | \alpha' j' m'). \qquad (33.1)$$

We apply (5.2) to the Clebsch-Gordan coefficient and (31.9) to the matrix elements. We sum over the three Wigner coefficients thus formed with the aid of (26.2) and obtain

$$\left(\alpha jm \left| \left[T^{k_1} \times U^{k_2} \right]^k_q \right| \alpha' j' m' \right) = (k)^{\frac{1}{2}} (-1)^{j+j'+k} \cdot (-1)^{j-m} \begin{pmatrix} j & k & j' \\ -m & q & m' \end{pmatrix} \times$$

$$\times \sum_{\alpha'' j''} (\alpha j \| T^{k_1} \| \alpha'' j'') (\alpha'' j'' \| U^{k_2} \| \alpha' j') \begin{Bmatrix} k_1 & k_2 & k \\ j' & j & j'' \end{Bmatrix}. \qquad (33.2)$$

Comparing (33.2) with (31.9), we obtain the formula for the reduced matrix element of the product

$$\left(\alpha j \left\| \left[T^{k_1} \times U^{k_2} \right]^k \right\| \alpha' j' \right) = (k)^{\frac{1}{2}} (-1)^{j+j'+k} \times$$

$$\times \sum_{\alpha'' j''} (\alpha j \| T^{k_1} \| \alpha'' j'') (\alpha'' j'' \| U^{k_2} \| \alpha' j') \begin{Bmatrix} k_1 & k_2 & k \\ j' & j & j'' \end{Bmatrix}. \qquad (33.3)$$

In particular, for the reduced matrix element of the scalar product we have

$$\left(\alpha j \| (T^k \cdot U^k) \| \alpha' j' \right) = \delta(j, j') (j)^{-\frac{1}{2}} \times$$

$$\times \sum_{\alpha'' j''} (-1)^{j-j''} (\alpha j \| T^k \| \alpha'' j'') (\alpha'' j'' \| U^k \| \alpha' j). \qquad (33.4)$$

Let us now turn to the tensor product (32.1) in which T^{k_1} acts on the coordinates 1 and U^{k_2} on the coordinates 2. In such cases the eigenfunctions αjm are usually constructed by coupling the angular momenta j_1 and j_2, the eigenfunctions of which depend on the corresponding coordinates. In this case

$$\left(\alpha_1 j_1 \alpha_2 j_2 jm \left| \left[T^{k_1} \times U^{k_2} \right]^k_q \right| \alpha_1' j_1' \alpha_2' j_2' j' m' \right) =$$

$$= \sum_{\substack{m_1 m_2 \\ q_1 q_2 \\ m_1' m_2'}} (j_1 j_2 jm | j_1 j_2 m_1 m_2) \times$$

$$\times (k_1 k_2 q_1 q_2 | k_1 k_2 kq) (j_1' j_2' m_1' m_2' | j_1' j_2' j' m') \times$$

$$\times (\alpha_1 j_1 m_1 | T^{k_1}_{q_1} | \alpha_1' j_1' m_1') (\alpha_2 j_2 m_2 | U^{k_2}_{q_2} | \alpha_2' j_2' m_2'). \qquad (33.5)$$

Proceeding as in (33.1), we obtain a sum of products of five Wigner coefficients. Summing it as in sections 26 and 27, we obtain

$$\left(\alpha_1 j_1 \alpha_2 j_2 jm \left| \left[T^{k_1} \times U^{k_2} \right]^k_q \right| \alpha_1' j_1' \alpha_2' j_2' j' m' \right) = \left[(k)(j)(j') \right]^{\frac{1}{2}} \times$$

$$\times (\alpha_1 j_1 \| T^{k'} \| \alpha_1' j_1') (\alpha_2 j_2 \| U^{k_2} \| \alpha_2' j_2') \times$$

$$\times (-1)^{j-m} \begin{pmatrix} j & k & j' \\ -m & q & m' \end{pmatrix} \cdot \begin{Bmatrix} j_1 & j_1' & k_1 \\ j_2 & j_2' & k_2 \\ j & j' & k \end{Bmatrix}. \qquad (33.6)$$

Comparing (33.6) and (31.9) we obtain the formula for the reduced matrix element of a tensor product the factors of which act on different coordinates

$$\left(\alpha_1 j_1 \alpha_2 j_2 j \left\| \left[T^{k_1} \times U^{k_2} \right]^k \right\| \alpha_1' j_1' \alpha_2' j_2' j' \right) = \left[(k)(j)(j') \right]^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_1' & k_1 \\ j_2 & j_2' & k_2 \\ j & j' & k \end{Bmatrix} \times$$

$$\times (\alpha_1 j_1 \| T^{k_1} \| \alpha_1' j_1') (\alpha_2 j_2 \| U^{k_2} \| \alpha_2' j_2'). \qquad (33.7)$$

For the scalar product ($k = 0$) the above formula can be simplified

$$\left(\alpha_1 j_1 \alpha_2 j_2 j \| (T^k \cdot U^k) \| \alpha_1' j_1' \alpha_2' j_2' j'\right) = (j)^{\frac{1}{2}} \delta(j, j') \cdot (-1)^{j_1' + j_1 + j} \times$$

$$\times \begin{Bmatrix} j_1 & j_2 & j \\ j_2' & j_1' & k \end{Bmatrix} \times (\alpha_1 j_1 \| T^{k_1} \| \alpha_1' j_1')(\alpha_2 j_2 \| U^{k_2} \| \alpha_2' j_2'). \qquad (33.8)$$

From (33.7) one can obtain convenient formulas for the reduced matrix element of the operator which acts only on the coordinate 1. To do this one must set $U^{k_2} = 1$, $k_2 = 0$, $k_1 = k$ and make use of (31.13). The result will be

$$(\alpha_1 j_1 \alpha_2 j_2 j \| T^k \| \alpha_1' j_1' \alpha_2' j_2' j') = \delta(\alpha_2, \alpha_2') \delta(j_2, j_2') \left[(j)(j')\right]^{\frac{1}{2}} \cdot (-1)^{j_1 + j_1 + j' + k} \times$$

$$\times \begin{Bmatrix} j_1 & j & j_2 \\ j' & j_1' & k \end{Bmatrix} (\alpha_1 j_1 \| T^k \| \alpha_1' j_1'). \qquad (33.9)$$

Analogously, for the operator which acts only on the coordinate 2 we will find

$$(\alpha_1 j_1 \alpha_2 j_2 j \| U^k \| \alpha_1' j_1' \alpha_2' j_2' j') = \delta(\alpha_1, \alpha_1') \delta(j_1, j_1') \left[(j)(j')\right]^{\frac{1}{2}} (-1)^{j_1 + j_1' + j + k} \times$$

$$\times \begin{Bmatrix} j_2 & j & j_1 \\ j' & j_2' & k \end{Bmatrix} (\alpha_2 j_2 \| U^k \| \alpha_2' j_2'). \qquad (33.10)$$

The last two formulas correspond to (44a, b) in Racah's work /1942/. In the first case the operator commutes with $\mathbf{J}_2$ and in the second case with $\mathbf{J}_1$.

The fundamental formulas of the algebra of tensor operators are (31.9), (33.3) and (33.7). All the rest are particular cases of these. When a large number of tensors are multiplied it is not expedient to look for general formulas for the matrix elements, as in such cases the calculation is carried out by repeated application of the formulas for the product of two tensors. Examples of such calculations are given in the next section.

34. Calculation of matrix elements of complex products of tensor operators

In this section we shall consider concrete examples of complex tensor products, that is to say, products of more than two tensors. We will choose examples which are characteristic of the calculation of matrix elements. We will therefore not be interested in the radial parts of the operators and will not write them explicitly.

Let us take a product of the following form

$$\left[\overset{i}{T}{}^{k_1} \times \overset{j}{U}{}^{k_2} \times \overset{l}{V}{}^{k_3} \times \overset{m}{W}{}^{k_4}\right]_q^k = \left[\left[\overset{i}{T}{}^{k_1} \times \overset{j}{U}{}^{k_2}\right]^{k_{12}} \times \left[\overset{l}{V}{}^{k_3} \times \overset{m}{W}{}^{k_4}\right]^{k_{34}}\right]_q^k. \qquad (34.1)$$

Here the supplementary superscripts i, j, l, m indicate the coordinates which are acted upon by the corresponding operators. We assume at first that $i = j = 1$, and

$l = m = 2$. Next using (33.7) and then (33.3) we obtain

$$\left(\alpha_1 j_1 \alpha_2 j_2 j \left\| \left[\left[\overset{1}{T}{}^{k_1} \times \overset{1}{U}{}^{k_2}\right]^{k_{12}} \times \left[\overset{2}{V}{}^{k_3} \times \overset{2}{W}{}^{k_4}\right]\right]^{k} \right\| \alpha_1' j_1' \alpha_2' j_2' j'\right) =$$

$$= \left[(k)(j)(j')(k_{12})(k_{34})\right]^{\frac{1}{2}} (-1)^{j_1 + j_1' + k_{12} + j_2 + j_2' + k_{34}} \times$$

$$\times \sum_{\substack{\alpha_1'' j_1'' \\ \alpha_1''' j_1'''}} (\alpha_1 j_1 \| \overset{1}{T}{}^{k_1} \| \alpha_1'' j_1'')(\alpha_1'' j_1'' \| \overset{1}{U}{}^{k_2} \| \alpha_1' j_1') \times$$

$$\times (\alpha_2 j_2 \| \overset{2}{V}{}^{k_3} \| \alpha_2'' j_2'')(\alpha_2'' j_2'' \| \overset{2}{W}{}^{k_4} \| \alpha_2' j_2') \times$$

$$\times \begin{Bmatrix} k_1 & k_2 & k_{12} \\ j_1' & j_1 & j_1'' \end{Bmatrix} \begin{Bmatrix} k_3 & k_4 & k_{34} \\ j_2' & j_2 & j_2'' \end{Bmatrix} \begin{Bmatrix} j_1 & j_1' & k_{12} \\ j_2 & j_2' & k_{34} \\ j & j' & k \end{Bmatrix}. \qquad (34.2)$$

If $i = l = 1$, and $j = m = 2$, then it is convenient to change the order of the multiplication of the tensors so that the operators which act on the same coordinates are multiplied immediately. With the aid of (23.7) we obtain

$$\left[\left[\overset{1}{T}{}^{k_1} \times \overset{2}{U}{}^{k_2}\right]^{k_{12}} \times \left[\overset{1}{V}{}^{k_3} \times \overset{2}{W}{}^{k_4}\right]^{k_{34}}\right]^{k} = \sum_{k_{13} k_{24}} \left[(k_{12})(k_{34})(k_{13})(k_{24})\right]^{\frac{1}{2}} \begin{Bmatrix} k_1 & k_2 & k_{12} \\ k_3 & k_4 & k_{34} \\ k_{13} & k_{24} & k \end{Bmatrix} \times$$

$$\times \left[\left[\overset{1}{T}{}^{k_1} \times \overset{1}{V}{}^{k_3}\right]^{k_{13}} \times \left[\overset{2}{U}{}^{k_2} \times \overset{2}{W}{}^{k_4}\right]^{k_{24}}\right]^{k}. \qquad (34.3)$$

To find the expression for the reduced matrix element of (34.3) we first make use of (33.7) and then apply (33.3) to the individual elements obtained. We find

$$\left(\alpha_1 j_1 \alpha_2 j_2 j \left\| \left[\left[\overset{1}{T}{}^{k_1} \times \overset{2}{U}{}^{k_2}\right]^{k_{12}} \times \left[\overset{1}{V}{}^{k_3} \times \overset{2}{W}{}^{k_4}\right]^{k_{34}}\right]^{k} \right\| \alpha_1' j_1' \alpha_2' j_2' j'\right) =$$

$$= \sum_{\substack{\alpha_1'' j_1'' \alpha_2'' j_2'' \\ k_{13} k_{24}}} (-1)^{j_1 + j_1' + k_{13} + j_2 + j_2' + k_{24}} \left[(k)(k_{12})(k_{34})(j)(j')\right]^{\frac{1}{2}} \times$$

$$\times (k_{13})(k_{24}) \begin{Bmatrix} k_1 & k_3 & k_{13} \\ j_1' & j_1 & j_1'' \end{Bmatrix} \begin{Bmatrix} k_2 & k_4 & k_{24} \\ j_2' & j_2 & j_2'' \end{Bmatrix} \times \begin{Bmatrix} k_1 & k_2 & k_{12} \\ k_3 & k_4 & k_{34} \\ k_{13} & k_{24} & k \end{Bmatrix} \begin{Bmatrix} j_1 & j_1' & k_{13} \\ j_2 & j_2' & k_{24} \\ j & j' & k \end{Bmatrix} \times$$

$$\times (\alpha_1 j_1 \| \overset{1}{T}{}^{k_1} \| \alpha_1'' j_1'')(\alpha_1'' j_1'' \| \overset{1}{V}{}^{k_3} \| \alpha_1' j_1') \times$$

$$\times (\alpha_2 j_2 \| \overset{2}{U}{}^{k_2} \| \alpha_2'' j_2'')(\alpha_2'' j_2'' \| \overset{2}{W}{}^{k_4} \| \alpha_2' j_2'). \qquad (34.4)$$

Summation over k_{13} and k_{24} by the graphical methods leads to the following result

$$\left(a_1 j_1\, a_2 j_2 j \left\| \left[\left[\overset{1}{\mathrm{T}}{}^{k_1} \times \overset{2}{\mathrm{U}}{}^{k_2}\right]^{k_{12}} \times \left[\overset{1}{\mathrm{V}}{}^{k_3} \times \overset{2}{\mathrm{W}}{}^{k_4}\right]^{k_{34}}\right]^{k} \right\| a_1' j_1'\, a_2' j_2' j'\right) =$$

$$= \sum_{a_1'' j_1'' a_2'' j_2''} (-1)^{j_1 + j_2 + j_1'' + j_2'' + k_1 + k_2} \left[(k)(k_{12})(k_{34})(j)(j')\right]^{\frac{1}{2}} \times$$

$$\times (a_1 j_1 \| \mathrm{T}^{k_1} \| a_1'' j_1'')(a_1'' j_1'' \| \mathrm{V}^{k_3} \| a_1' j_1')(a_2 j_2 \| \mathrm{U}^{k_2} \| a_2'' j_2'')(a_2'' j_2'' \| \mathrm{W}^{k_4} \| a_2' j_2') \times$$

$$\times \begin{Bmatrix} j_1 & j_1' & j & j' & j_2 & j_2' \\ j_1'' & & k & & j_2'' & \\ k_1 & k_2 & k_{12} & k_{34} & k_3 & k_4 \end{Bmatrix}. \qquad (34\,5)$$

In the numerical evaluation of this sum one should make use of the expansion (20.3) for the $15j$-coefficient of the third kind.

If in (34.1) $i = j = l = 1$ and $m = 2$, then the product should be transformed as follows

$$\left[\left[\overset{1}{\mathrm{T}}{}^{k_1} \times \overset{1}{\mathrm{U}}{}^{k_2}\right]^{k_{12}} \times \left[\overset{1}{\mathrm{V}}{}^{k_3} \times \overset{2}{\mathrm{W}}{}^{k_4}\right]^{k_{34}}\right]^{k} = \sum_{k_{123}} (-1)^{k_3 + k_4 + k_{12} + k} \left[(k_{34})(k_{123})\right]^{\frac{1}{2}} \times$$

$$\begin{Bmatrix} k_3 & k_4 & k_{34} \\ k & k_{12} & k_{123} \end{Bmatrix} \left[\left[\left[\overset{1}{\mathrm{T}}{}^{k_1} \times \overset{1}{\mathrm{U}}{}^{k_2}\right]^{k_{12}} \times \overset{1}{\mathrm{V}}{}^{k_3}\right]^{k_{123}} \times \overset{2}{\mathrm{W}}{}^{k_4}\right]^{k}. \qquad (34.6)$$

Here the transformation matrix is simplified in accordance with (9.2) and expressed in terms of the $6j$-coefficient. Applying (33.7) once and (33.3) twice we find

$$\left(a_1 j_1\, a_2 j_2 j \left\| \left[\left[\overset{1}{\mathrm{T}}{}^{k_1} \times \overset{1}{\mathrm{U}}{}^{k_2}\right]^{k_{12}} \times \left[\overset{1}{\mathrm{V}}{}^{k_3} \times \overset{2}{\mathrm{W}}{}^{k_4}\right]^{k_{34}}\right]^{k} \right\| a_1' j_1'\, a_2' j_2' j'\right) =$$

$$= \sum_{a_1'' j_1''\, a_1''' j_1'''\, k_{123}} (-1)^{k_3 + k_4 + k_{12} + k + j_1 + j_1' + k_{123} + j_1''' + j_1 + k_{12}} \times$$

$$\times \left[(k)(k_{12})(k_{34})(j)(j')\right]^{\frac{1}{2}} (k_{123}) \times$$

$$\times \begin{Bmatrix} k_3 & k_4 & k_{34} \\ k & k_{12} & k_{123} \end{Bmatrix} \begin{Bmatrix} k_{12} & k_3 & k_{123} \\ j_1' & j_1 & j_1''' \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & k_{12} \\ j_1''' & j_1 & j_1'' \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} j_1 & j_1' & k_{123} \\ j_2 & j_2' & k_4 \\ j & j' & k \end{Bmatrix} \times$$

$$\times (a_1 j_1 \| \mathrm{T}^{k_1} \| a_1'' j_1'')(a_1'' j_1'' \| \mathrm{U}^{k_2} \| a_1''' j_1''') \times$$

$$\times (a_1''' j_1''' \| \mathrm{V}^{k_3} \| a_1' j_1')(a_2 j_2 \| \mathrm{W}^{k_4} \| a_2' j_2'). \qquad (34.7)$$

Summing over k_{123}, we obtain

$$\left(\alpha_1 j_1 \alpha_2 j_2 j \left\| \left[\left[\overset{1}{\mathrm{T}}{}^{k_1} \times \overset{1}{\mathrm{U}}{}^{k_2} \right]^{k_{12}} \times \left[\overset{1}{\mathrm{V}}{}^{k_3} \times \overset{2}{\mathrm{W}}{}^{k_4} \right]^{k_{34}} \right]^{k} \right\| \alpha_1' j_1' \alpha_2' j_2' j' \right) =$$

$$= (-1)^{j_1' - j_2 + j_1'' + j - j' - k_3 - k_4 + k} \left[(k)(k_{12})(k_{34})(j)(j') \right]^{\frac{1}{2}} \times$$

$$\times \sum_{\alpha_1'' j_1'' \alpha_1''' j_1'''} (-1)^{j_1'''} \begin{Bmatrix} k_1 & k_2 & k_{12} \\ j_1''' & j_1 & j_1'' \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} k_{34} & k_4 & j_2 & j_1 \\ k_3 & j_2' & j & k_{12} \\ j_1''' & j_1' & j' & k \end{Bmatrix} \times$$

$$\times (\alpha_1 j_1 \| \mathrm{T}^{k_1} \| \alpha_1'' j_1'') (\alpha_1'' j_1'' \| \mathrm{U}^{k_2} \| \alpha_1''' j_1''') \times$$

$$\times (\alpha_1''' j_1''' \| \mathrm{V}^{k_3} \| \alpha_1' j_1') (\alpha_2 j_2 \| \mathrm{W}^{k_4} \| \alpha_2' j_2'). \qquad (34.8)$$

In the phase factors of the above formulas we took into account the fact that the ranks of the tensors are integers.

35. Double tensors, their products and matrix elements

In many cases it is convenient to introduce the irreducible double tensors $\mathrm{T}^{k_1 k_2}$ with components $\mathrm{T}^{k_1 k_2}_{q_1 q_2}$ $(q_1 = -k_1, \ldots, k_1;\ q_2 = -k_2, \ldots, k_2)$. The $2k_1 + 1$ components of this tensor with fixed q_2 and different q_1 form the basis of the irreducible representation $\mathcal{D}_{k_1}$ upon rotation of one space, the $2k_2 + 1$ components with fixed q_1 and different q_2 transform according to $\mathcal{D}_{k_2}$ upon rotation of another space. For the sake of brevity we say that $\mathrm{T}^{k_1 k_2}$ is an irreducible tensor of rank k_1 with respect to $\mathbf{J}_1$ and an irreducible tensor of rank k_2 with respect to $\mathbf{J}_2$. k_1 and k_2 are usually integers. Instead of (31.9) we have for this tensor

$$\left(\alpha j_1 j_2 m_1 m_2 \left| \mathrm{T}^{k_1 k_2}_{q_1 q_2} \right| \alpha' j_1' j_2' m_1' m_2'\right) = (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & k_1 & j_1' \\ -m_1 & q_1 & m_1' \end{pmatrix} \times$$

$$\times (-1)^{j_2 - m_2} \begin{pmatrix} j_2 & k_2 & j_2' \\ -m_2 & q_2 & m_2' \end{pmatrix} \times$$

$$\times (\alpha j_1 j_2 \| \mathrm{T}^{k_1 k_2} \| \alpha' j_1' j_2'). \qquad (35.1)$$

Upon simultaneous rotation of the two spaces, all the $(2k_1 + 1)(2k_2 + 1)$ components $\mathrm{T}^{k_1 k_2}_{q_1 q_2}$ transform according to the reducible representation $\mathcal{D}_{k_1} \times \mathcal{D}_{k_2}$. Therefore a tensor which is irreducible with respect to $\mathbf{J}_1$ and $\mathbf{J}_2$ is reducible with respect to

$\mathbf{J}=\mathbf{J}_1+\mathbf{J}_2$. If we reduce $\mathcal{D}_{k_1}\times\mathcal{D}_{k_2}$, i.e., pass to the quantitites

$$T^{k_1k_2k}_{q}=\sum_{q_1q_2}T^{k_1k_2}_{q_1q_2}(k_1k_2q_1q_2\,|\,k_1k_2kq), \tag{35.2}$$

the double tensor $T^{k_1k_2}$ becomes the set of double tensors $T^{k_1k_2k}$ of ranks $k=|k_1-k_2|$, ..., $|k_1+k_2|$, reduced with respect to $\mathbf{J}$.

Let us obtain the formula which relates the reduced matrix element of the reduced double tensor to the reduced matrix element of the non-reduced double tensor which appears in (35.1). As (35.1) is analogous to (32.1), proceeding literally in the same way as in the derivation of (33.7) we find

$$(\alpha j_1j_2j\|\,T^{k_1k_2k}\,\|\alpha'j_1'j_2'j')=\left[(j)(j')(k)\right]^{\frac{1}{2}}\begin{Bmatrix}j_1 & j_2 & j\\ j_1' & j_2' & j'\\ k_1 & k_2 & k\end{Bmatrix}\times$$
$$\times(\alpha j_1j_2\|\,T^{k_1k_2}\,\|\alpha'j_1'j_2'). \tag{35.3}$$

The similarity of (35.3) and (33.7) is due to the fact that we may, if we wish, regard $[T^{k_1}\times U^{k_2}]^k$ as a reduced double tensor of rank k_1 with respect to $\mathbf{J}_1$ and k_2 with respect to $\mathbf{J}_2$. The non-reduced double tensor in this case is the indefinite (dyadic) product $T^{k_1}\times U^{k_2}$ with the components $T^{k_1}_{q_1}\cdot U^{k_2}_{q_2}$. The internal structure of double tensors can be more general than that of tensor products. Thus, it may be a linear combination of such tensor products.

The tensor product of two irreducible double tensors is defined in the same way as in the case of the usual tensors. By analogy with (32.1) we write

$$\left[T^{k_1k_2k}\times U^{k_1'k_2'k'}\right]^K_Q=\sum_{qq'}T^{k_1k_2k}_{q}U^{k_1'k_2'k'}_{q'}(kk'qq'\,|\,kk'KQ). \tag{35.4}$$

To find the reduced matrix element of the product (35.4) we make use of (33.3) and apply (35.3) to the individual reduced matrix elements obtained. We then find

$$\left(\alpha j_1j_2j\left\|\left[T^{k_1k_2k}\times U^{k_1'k_2'k'}\right]^K\right\|\alpha'j_1'j_2'j'\right)=$$
$$=\left[(k)(k')(K)(j)(j')\right]^{\frac{1}{2}}\times$$
$$\times\sum_{\alpha''j_1''j_2''j''}(-1)^{j+j'+K}(j'')(\alpha j_1j_2\|\,T^{k_1k_2}\,\|\alpha''j_1''j_2'')\times$$
$$\times(\alpha''j''j_2''\|\,U^{k_1'k_2'}\,\|\alpha'j_1'j_2')\times$$
$$\times\begin{Bmatrix}k & k' & K\\ j' & j & j''\end{Bmatrix}\begin{Bmatrix}j_1 & j_1'' & k_1\\ j_2 & j_2'' & k_2\\ j & j'' & k\end{Bmatrix}\begin{Bmatrix}j_1'' & j_1' & k_1'\\ j_2'' & j_2' & k_2'\\ j'' & j' & k'\end{Bmatrix}. \tag{35.5}$$

In accordance with the expression (20.3), for the 15j-coefficient the above formula

can be written as follows

$$\left(\alpha j_1 j_2 j \left\| \left[T^{k_1 k_2 k} \times U^{k_1' k_2' k'} \right]^K \right\| \alpha' j_1' j_2' j' \right) =$$

$$= \left[(k)(k')(K)(j)(j') \right]^{\frac{1}{2}} (-1)^{j_1' + j_2' + k_1' + k_2'} \times$$

$$\times \sum_{\alpha'' j_1'' j_2''} (-1)^{j_1'' + j_2''} (\alpha j_1 j_2 \| T^{k_1 k_2} \| \alpha'' j_1'' j_2'') \times$$

$$\times (\alpha'' j_1'' j_2'' \| U^{k_1' k_2'} \| \alpha' j_1' j_2') \times$$

$$\times \begin{Bmatrix} k_1 & k_1' & k & k' & k_2 & k_2' \\ j_1'' & & K & & j_2'' & \\ j_1 & j_1' & j & j' & j_2 & j_2' \end{Bmatrix}. \qquad (35.5)$$

This formula is the analogue of (33.7) for double tensors. If in (35.4) $K = 0$, then, taking (32.5), (16.5) and (28.1) into account, we obtain the following expression from (35.5)

$$\left(\alpha j_1 j_2 j \| (T^{k_1 k_2 k} \cdot U^{k_1' k_2' k'}) \| \alpha' j_1' j_2' j' \right) =$$

$$= \delta(jj') (k)(j)^{\frac{1}{2}} (-1)^{j - j_1' - j_2' + k_1' + k_2'} \times$$

$$\times \sum_{\alpha'' j_1'' j_2''} (\alpha j_1 j_2 \| T^{k_1 k_2} \| \alpha'' j_1'' j_2'') (\alpha'' j_1'' j_2'' \| U^{k_1' k_2'} \| \alpha' j_1' j_2') \times$$

$$\times \begin{bmatrix} k & j_2'' & j_1'' & j \\ k_1 & k_1' & j_2 & j_2' \\ k_2 & j_1 & k_2' & j_1' \end{bmatrix}. \qquad (35.7)$$

This formula gives the expression for the reduced matrix element of the scalar product of two double tensors.

Appendix 1

Notations for the Wigner, $6j$-, $9j$- and allied coefficients

In this appendix we give the notations for the Clebsch-Gordan coefficients and the notations and forms of the Wigner, $6j$- and $9j$-coefficients used by various authors.

Notations for the Clebsch-Gordan coefficients

Condon and Shortley /1949/, Racah /1942/, et al

$$(j_1 j_2 jm \mid j_1 j_2 m_1 m_2) = (j_1 j_2 m_1 m_2 \mid j_1 j_2 jm) =$$
$$= (jm \mid m_1 m_2) = (m_1 m_2 \mid jm);$$

Wigner /1931/

$$s^{j_1 j_2}_{jm_1 m_2};$$

Fok /1940/

$$c^{j_1 j_2}_{jm}(m_1, m_2);$$

Van der Waerden /1938/, Landau and Lifshits /1948/

$$C^{j}_{m_1 m_2};$$

Boys and Sahni /1954/

$$X(j, m, j_1, j_2, m_1);$$

Alder /1952/

$$c^{jm}_{j_1 m_1\ j_2 m_2};$$

Rose /1957/

$$C(j_1 j_2 j, m_1 m_2);$$

Fano /1951/

$$\left\langle j_1 m_1, j_2 m_2 \mid (j_1 j_2) jm \right\rangle;$$

Biedenharn /1952/, Redmond /1954/

$$C^{j_1 j_2 j}_{m_1 m_2 m}.$$

Notations and forms for the Wigner coefficients

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \qquad \text{(A.1.1)}$$

Racah /1942/

$$= (-1)^{-j_1+j_2+j_3} V(j_1 j_2 j_3, m_1 m_2 m_3) \qquad \text{(a)}$$

Landau and Lifshits /1948/

$$= (-1)^{j_1-j_2+j_3} S_{j_1 m_1, j_2 m_2, j_3 m_3} \qquad \text{(b)}$$

Fano /1951/

$$= (-1)^{j_1-j_2+j_3} \left\langle j_1 m_1, j_2 m_2, j_3 m_3 \mid 0 \right\rangle \qquad \text{(c)}$$

Schwinger /1952/

$$= X(j_1 j_2 j_3,\ m_1 m_2 m_3)\cdot \qquad \text{(d)}$$

Notations and forms for the $6j$-coefficients (Racah coefficients)

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{matrix} \right\} = \qquad \text{(A.1.2)}$$

Racah /1942/

$$= (-1)^{j_1+j_2+l_1+l_2} W(j_1 j_2 l_2 l_1;\ j_3 l_3); \qquad \text{(a)}$$

Jahn /1951/

$$= (-1)^{j_1+j_2+l_1+l_2} \left[(j_3)(l_3)\right]^{-\frac{1}{2}} U(j_1 j_2 l_2 l_1;\ j_3 l_3); \qquad \text{(b)}$$

Boys and Sahni /1954/

$$= \left[(j_3)(l_3)\right]^{-\frac{1}{2}} U\begin{pmatrix} j_1 j_2 l_1 l_2 \\ j_3 l_3 \end{pmatrix}; \qquad \text{(c)}$$

Banerjee and Saha /1954/

$$= (-1)^{j_1+j_2+l_1+l_2} \begin{bmatrix} j_1 & j_2 & \\ l_2 & l_1 & j_3 \\ & l_3 & \end{bmatrix}. \qquad \text{(d)}$$

The quantities $\mathbf{U}$ are the transformation matrices for three angular momenta (see (23.1)).

Notations and forms for the $9j$-coefficient (Wigner's $9j$-coefficient)

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{matrix} \right\} = \qquad \text{(A.1.3)}$$

Fano /1951/

$$= X\begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{pmatrix}; \qquad \text{(a)}$$

Schwinger /1952/

$$= (-1)^{j_1+k_2-j_3-l_2} S(j_1 j_2 k_1 k_2;\ j_3 k_3 l_1 l_2;\ l_3); \qquad \text{(b)}$$

Hope /Jahn and Hope 1954/

$$= \left[(j_3)(k_3)(l_1)(l_2)\right]^{-\frac{1}{2}} \chi(j_1 j_2 k_1 k_2;\ j_3 k_3;\ l_1 l_2;\ l_3); \qquad \text{(c)}$$

Kennedy and Cliff /1957/

$$= \left[(j_3)(k_3)(l_1)(l_2)\right]^{-\frac{1}{2}} A\begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{pmatrix}; \qquad \text{(d)}$$

Arima, Horie and Tanabe /1954/

$$= \mathbf{U}\begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{pmatrix}. \qquad \text{(e)}$$

The quantities χ and A are the transformation matrices for four angular momenta (see (23.4) and (23.7)).

The notation for the $12j$-coefficient of the first kind is universal (formula (19.1)). The various notations for the $12j$-coefficient of the second kind are given in formulas (19.3)-(19.5).

Appendix 2

Algebraic formulas for the Clebsch-Gordan coefficients

In this section we give the algebraic formulas for the coefficient $B_{j_1k,m_1}(j_1,m)$ which enters into the formulas for the Clebsch-Gordan coefficients as follows (see the end of section 4)

$$(j_1j_2m-m_2m_2|j_1j_2j_1+k\ m)=A_{j_1k}(j_1)\cdot B_{j_1k,m_1}(j_1,m), \tag{A.2.1}$$

where*

$$A_{j_1k}(j_1)=\left[\frac{(2j_1+j_2+k+1)^{(2j_1+1)}}{2j_1+2k+1}\right]^{-\frac{1}{2}}, \tag{A.2.2}$$

$$U^{(x)}=U(U-1)\ldots(U-x+1). \tag{A.2.3}$$

The coefficient B satisfies the following condition

$$B_{j_1k,-m_1}(j_1,m)=(-1)^{j_1-k}B_{j_1k,m_1}(j_1,-m),$$

The formulas are therefore given only for non-negative values of m_2.

$$j_2=\frac{1}{2}$$

k \ m_2	$\frac{1}{2}$
$-\frac{1}{2}$	$-\left[j_1-m+\frac{1}{2}\right]^{\frac{1}{2}}$
$\frac{1}{2}$	$\left[j_1+m+\frac{1}{2}\right]^{\frac{1}{2}}$

---

* See formula (4.17)—Translator.

$j_2 = 1$

k \ m_2	0	1
-1	$-\left[2(j_1+m)(j_1-m)\right]^{\frac{1}{2}}$	$\left[(j_1-m)(j_1-m+1)\right]^{\frac{1}{2}}$
0	$2m$	$-\left[2(j_1+m)(j_1-m+1)\right]^{\frac{1}{2}}$
1	$\left[2(j_1+m+1)(j_1-m+1)\right]^{\frac{1}{2}}$	$\left[(j_1+m)(j_1+m+1)\right]^{\frac{1}{2}}$

$j_2 = \frac{3}{2}$

k \ m_2	$\frac{1}{2}$	$\frac{3}{2}$
$-\frac{3}{2}$	$\left[3\left(j_1+m-\frac{1}{2}\right)\left(j_1-m+\frac{1}{2}\right)^{(2)}\right]^{\frac{1}{2}}$	$-\left[\left(j_1-m+\frac{3}{2}\right)^{(3)}\right]^{\frac{1}{2}}$
$-\frac{1}{2}$	$-\left(j_1+3m-\frac{1}{2}\right)\left[j_1-m+\frac{1}{2}\right]^{\frac{1}{2}}$	$\left[3\left(j_1+m-\frac{1}{2}\right)\left(j_1-m+\frac{3}{2}\right)^{(2)}\right]^{\frac{1}{2}}$
$\frac{1}{2}$	$-\left(j_1-3m+\frac{3}{2}\right)\left[j_1+m+\frac{1}{2}\right]^{\frac{1}{2}}$	$-\left[3\left(j_1+m+\frac{1}{2}\right)^{(2)}\left(j_1-m+\frac{3}{2}\right)\right]^{\frac{1}{2}}$
$\frac{3}{2}$	$\left[3\left(j_1+m+\frac{3}{2}\right)^{(2)}\left(j_1-m+\frac{3}{2}\right)\right]^{\frac{1}{2}}$	$\left[\left(j_1+m+\frac{3}{2}\right)^{(3)}\right]^{\frac{1}{2}}$

$j_2 = 2$

m_2 / k	0	1	2
−2	$\left[6(j_1+m)^{(2)}(j_1-m)^{(2)}\right]^{\frac{1}{2}}$	$-2\left[(j_1+m-1)(j_1-m+1)^{(2)}\right]^{\frac{1}{2}}$	$\left[(j_1-m+2)^{(4)}\right]^{\frac{1}{2}}$
−1	$-2m\left[6(j_1+m)(j_1-m)\right]^{\frac{1}{2}}$	$2(j_1+2m-1)\left[(j_1-m+1)^{(2)}\right]^{\frac{1}{2}}$	$-2\left[(j_1+m-1)(j_1-m+2)^{(2)}\right]^{\frac{1}{2}}$
0	$2\left[3m^2-j_1(j_1+1)\right]$	$-(2m-1)\left[6(j_1+m)(j_1-m+1)\right]^{\frac{1}{2}}$	$\left[6(j_1+m)^{(2)}(j_1-m+2)^{(2)}\right]^{\frac{1}{2}}$
1	$2m\left[6(j_1+m+1)(j_1-m+1)\right]^{\frac{1}{2}}$	$-2(j_1-2m+2)\left[(j_1+m+1)^{(2)}\right]^{\frac{1}{2}}$	$-2\left[(j_1+m+1)^{(2)}(j_1-m+2)\right]^{\frac{1}{2}}$
2	$\left[6(j_1+m+2)^{(2)}(j_1-m+2)^{(2)}\right]^{\frac{1}{2}}$	$2\left[(j_1+m+2)^{(2)}(j_1-m+2)\right]^{\frac{1}{2}}$	$\left[(j_1+m+2)^{(4)}\right]^{\frac{1}{2}}$

Note

(Tables for $j_2 = 5/2$, 3, 7/2 and 4 are given separately at the end of the book

Appendix 3

Diagrams of 18 j-coefficients

In this appendix we give the (conventionally) standard forms of the two-dimensional diagrams of those 18 j-coefficients which do not decompose into simpler j-coefficients. There are 18 of these. They are denoted by capital letters corresponding to the algebraic formulas of the next appendix.

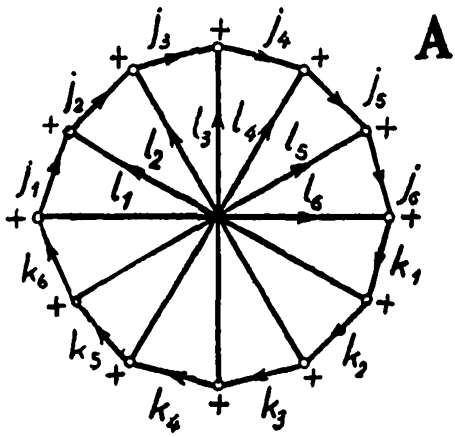

Figure A. 3.1

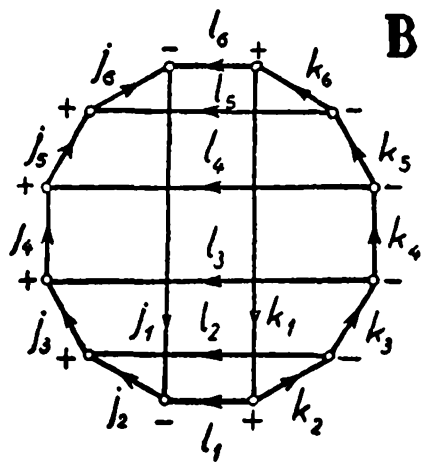

Figure A. 3.2

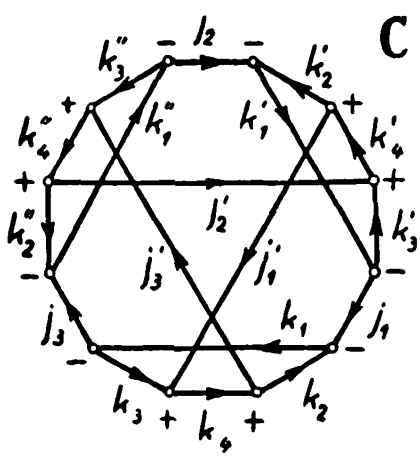

Figure A. 3.3

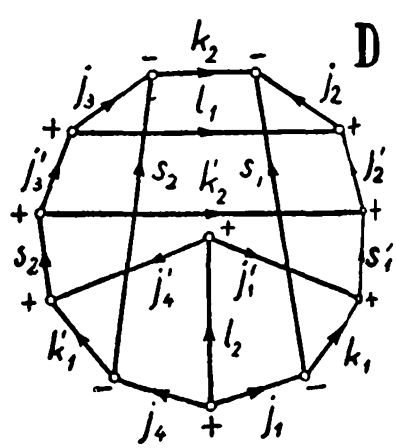

Figure A. 3.4

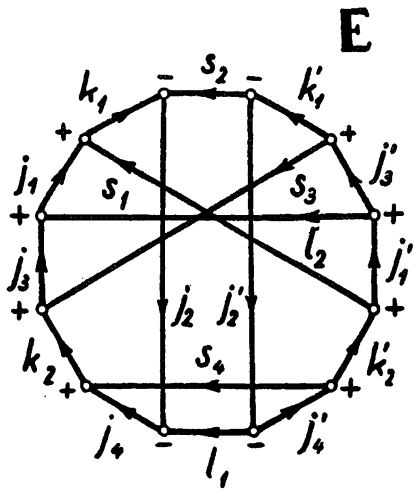

Figure A. 3.5

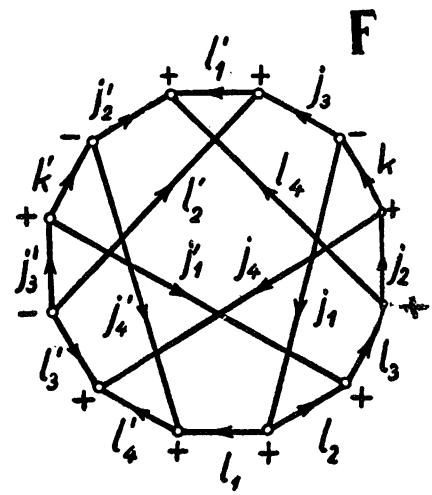

Figure A. 3.6

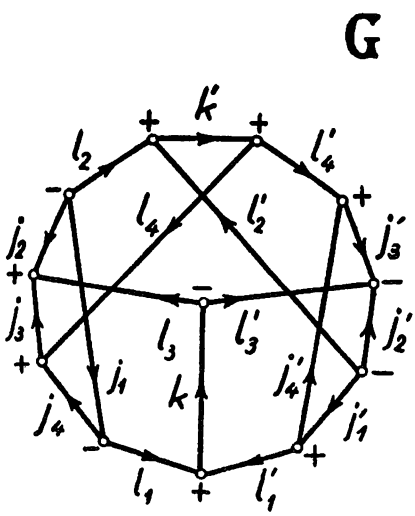

Figure A. 3.7

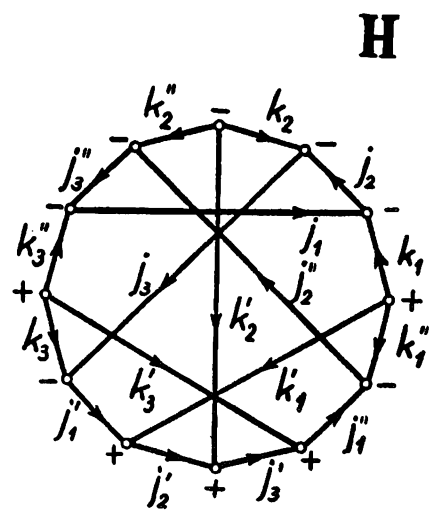

Figure A. 3.8

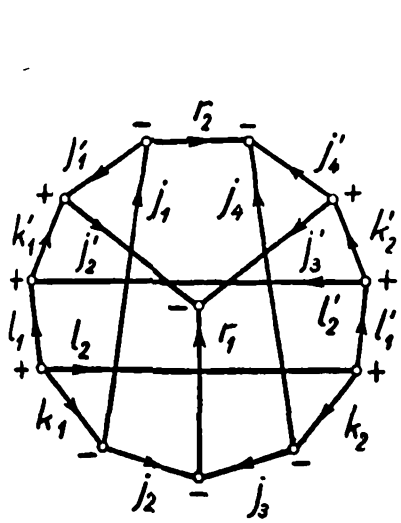

Figure A. 3.9

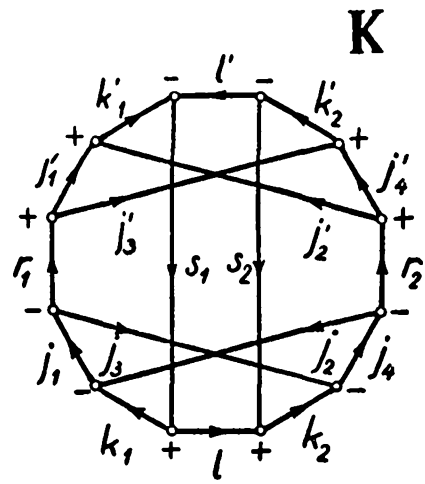

Figure A. 3.10

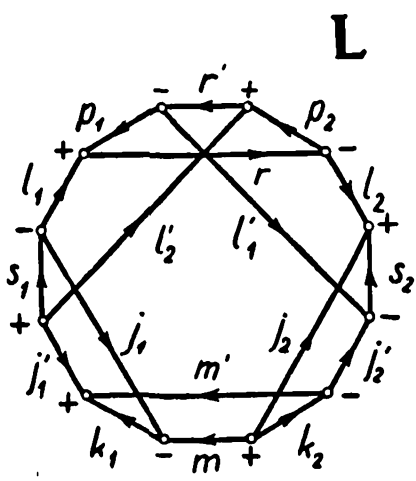

Figure A. 3.12

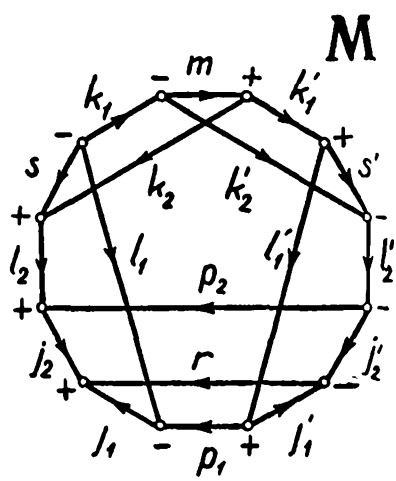

Figure A. 3.13

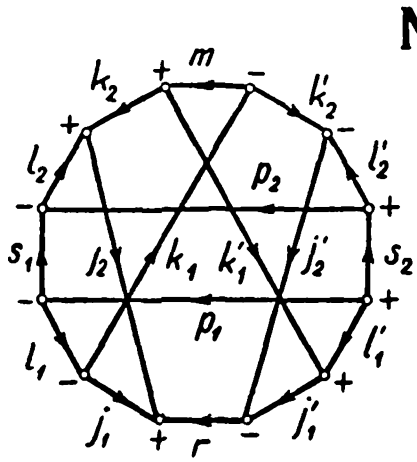

Figure A. 3.15

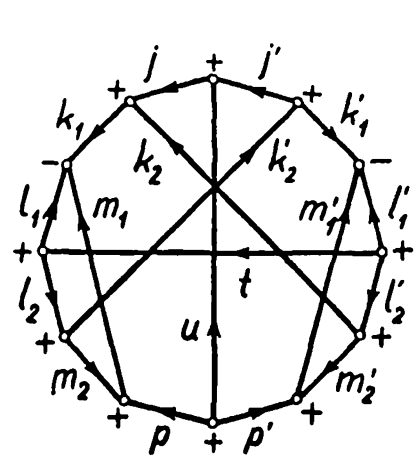

Figure A. 3.16

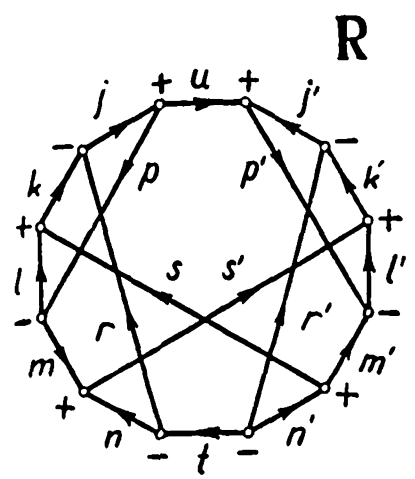

Figure A. 3.17

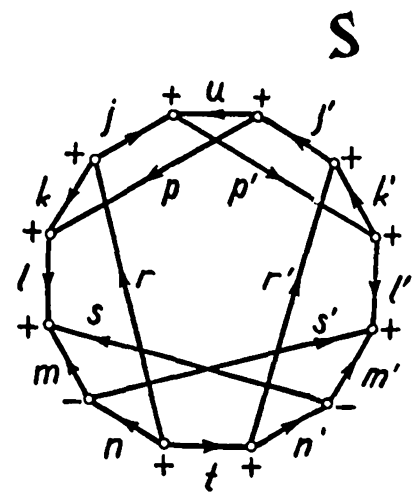

Figure A. 3.18

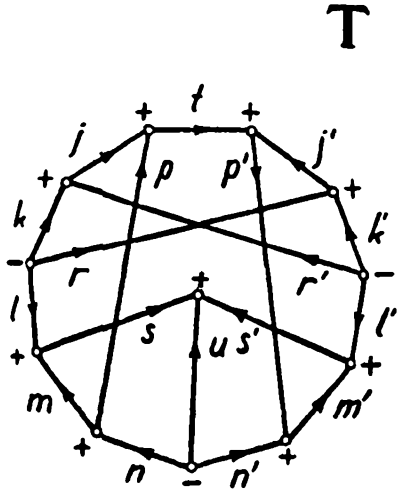

Figure A. 3.11

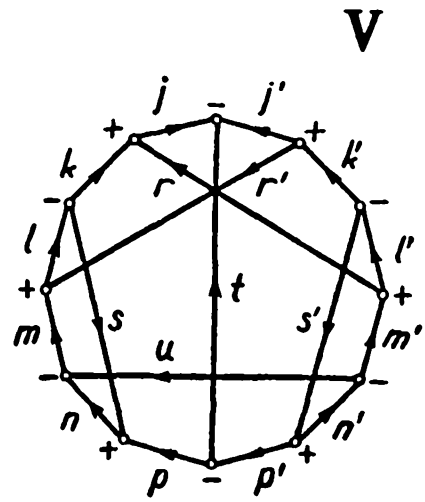

Figure A. 3.14

Appendix 4

Properties of 18j-coefficients

In this appendix we give the symmetry properties of the 18j-coefficients and their expressions in terms of simpler j-coefficients. The letters on the left-hand side of each equation indicate the corresponding diagrams of the 18j-coefficients in appendix 3.

$$A\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{Bmatrix} = \begin{Bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \end{Bmatrix} =$$

$$= \begin{Bmatrix} j_3 & j_3 & j_4 & j_5 & j_6 & k_1 \\ l_2 & l_3 & l_4 & l_5 & l_6 & l_1 \\ k_2 & k_3 & k_4 & k_5 & k_6 & j_1 \end{Bmatrix} = \begin{Bmatrix} k_1 & j_6 & j_5 & j_4 & j_3 & j_2 \\ l_6 & l_5 & l_4 & l_3 & l_2 & l_1 \\ j_1 & k_6 & k_5 & k_4 & k_3 & k_2 \end{Bmatrix} =$$

$$= \sum_x (x)(-1)^{R_4+x} \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{Bmatrix} \begin{Bmatrix} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} j_4 & k_4 & x \\ k_5 & j_5 & l_4 \end{Bmatrix} \begin{Bmatrix} j_5 & k_5 & x \\ k_6 & j_6 & l_5 \end{Bmatrix} \begin{Bmatrix} j_6 & k_6 & x \\ j_1 & k_1 & l_6 \end{Bmatrix}. \qquad \text{(A.4.1)}$$

$$B\begin{bmatrix} j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{bmatrix}=\begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \end{bmatrix}=$$

$$=\begin{bmatrix} j_2 & j_3 & j_4 & j_5 & j_6 & j_1 \\ l_2 & l_3 & l_4 & l_5 & l_6 & l_1 \\ k_2 & k_3 & k_4 & k_5 & k_6 & k_1 \end{bmatrix}=\begin{bmatrix} j_1 & j_6 & j_5 & j_4 & j_3 & j_2 \\ l_6 & l_5 & l_4 & l_3 & l_2 & l_1 \\ k_1 & k_6 & k_5 & k_4 & k_3 & k_2 \end{bmatrix}=$$

$$=\sum_x (x)(-1)^{R_4+2x}\begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix}\begin{Bmatrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{Bmatrix}\begin{Bmatrix} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{Bmatrix}\times$$

$$\times\begin{Bmatrix} j_4 & k_4 & x \\ k_5 & j_5 & l_4 \end{Bmatrix}\begin{Bmatrix} j_5 & k_5 & x \\ k_6 & j_6 & l_5 \end{Bmatrix}\begin{Bmatrix} j_6 & k_6 & x \\ k_1 & j_1 & l_6 \end{Bmatrix}. \qquad \text{(A.4.2)}$$

$$C\qquad \begin{Bmatrix} j_1 & k_1 & k_2 & k_3 & k_4 & j_1' \\ j_2 & k_1' & k_2' & k_3' & k_4' & j_2'' \\ j_3 & k_1'' & k_2'' & k_3'' & k_4'' & j_3' \end{Bmatrix}=$$

$$=\begin{Bmatrix} j_1' & k_4 & k_3 & k_2 & k_1 & j_1 \\ j_2' & k_4' & k_3' & k_2' & k_1' & j_2 \\ j_3' & k_4'' & k_3'' & k_2'' & k_1'' & j_3 \end{Bmatrix}=\begin{Bmatrix} j_3 & k_1'' & k_2'' & k_3'' & k_4'' & j_3' \\ j_1 & k_1 & k_2 & k_3 & k_4 & j_1' \\ j_2 & k_1' & k_2' & k_3' & k_4' & j_2'' \end{Bmatrix}=$$

$$=\begin{Bmatrix} j_1 & k_1 & k_2 & k_3 & k_4 & j_1' \\ j_3 & k_1'' & k_2'' & k_3'' & k_4'' & j_3' \\ j_2 & k_1' & k_2' & k_3' & k_4' & j_2'' \end{Bmatrix}=(-1)^{\varphi}\begin{Bmatrix} j_1' & k_3 & k_4 & k_1 & k_2 & j_1 \\ j_2 & k_2' & k_1' & k_4' & k_3' & j_2'' \\ j_3 & k_1'' & k_2'' & k_3'' & k_4'' & j_3'' \end{Bmatrix}=$$

$$=\sum_x (x)(-1)^{\psi}\begin{Bmatrix} j_1 & x & j_1' \\ k_3' & j_2'' & k_4' \\ k_1' & j_3 & k_2'' \end{Bmatrix}\begin{Bmatrix} j_2 & x & j_2'' \\ k_3'' & j_3' & k_4'' \\ k_1'' & j_3 & k_2'' \end{Bmatrix}\begin{Bmatrix} j_3 & x & j_3' \\ k_3 & j_1' & k_4 \\ k_1 & j_1 & k_2 \end{Bmatrix}. \qquad \text{(A.4.3)}$$

$$\varphi=j_2-j_3+j_3'-j_3',$$

$$\psi=\sum_{i=1}^{4}(j_i'-j_i)+k_4+k_4'+k_4''-k_1-k_1'-k_1''+2x.$$

$$D \quad \begin{Vmatrix} j_1 & j_2 & l_1 & j_3 & j_4 \\ & s_1 & & k_2 & s_2 \\ k_1 & & & & k_1' \\ & s_1' & & k_2' & s_2' \\ j_1'' & j_2'' & l_2 & j_3'' & j_4'' \end{Vmatrix} =$$

$$= \begin{Vmatrix} j_4 & j_3 & l_1 & j_2 & j_1 \\ & s_2 & & k_2 & s_1 \\ k_1' & & & & k_1 \\ & s_2' & & k_2' & s_1' \\ j_4'' & j_3'' & l_2 & j_2'' & j_1'' \end{Vmatrix} = \begin{Vmatrix} j_2 & j_1 & l_2 & j_1'' & j_2'' \\ & s_1 & & k_1 & s_1' \\ k_2 & & & & k_2' \\ & s_2 & & k_1' & s_2' \\ j_3 & j_4 & l_1 & j_4'' & j_3'' \end{Vmatrix} =$$

$$= \sum_x (x)(-1)^{\psi} \begin{Bmatrix} j_2 j_3 x \\ s_2 s_1 k_2 \end{Bmatrix} \begin{Bmatrix} j_2' j_3' x \\ s_2' s_1' k_2' \end{Bmatrix} \begin{Bmatrix} j_2 j_2'' l_1 \\ j_3'' j_3 x \end{Bmatrix} \times$$

$$\times \begin{bmatrix} k_1' l_2 x k_1 \\ s_2 s_2' j_1 j_1'' \\ j_4 s_1 j_4'' s_1' \end{bmatrix}. \qquad \text{(A.4.4)}$$

$$\psi = j_1 - j_2 + j_3 + j_4 + j_1'' + j_2' - j_3' + j_4'' + l_2 + k_1 + k_1' + x.$$

$$E \begin{Vmatrix} j_1 & j_2 & l_1 & j_3 & j_4 \\ & k_1 & & & k_2 \\ s_1 & s_2 & & s_3 & s_4 \\ & & k_1' & & & k_2' \\ j_1'' & j_2'' & l_2 & j_3'' & j_4'' \end{Vmatrix} = \begin{Vmatrix} j_3'' & j_2'' & l_1 & j_1'' & j_4'' \\ & k_1' & & & k_2' \\ s_3 & s_2 & & s_1 & s_4 \\ & & k_1 & & & k_2 \\ j_3 & j_2 & l_2 & j_1 & j_4 \end{Vmatrix} =$$

$$= (-1)^{\varphi} \begin{Vmatrix} j_2 & j_1 & l_2 & j_4 & j_3 \\ & k_1 & & & k_2 \\ s_2 & s_1 & & s_4 & s_3 \\ & & k_2' & & & k_1' \\ j_2'' & j_1'' & l_1 & j_4'' & j_3'' \end{Vmatrix} =$$

$$= \sum_x (x)(-1)^{\psi} \begin{Bmatrix} k_1 k_1' x \\ j_2' j_2 s_2 \end{Bmatrix} \begin{Bmatrix} k_2 k_2' x \\ j_4'' j_4 s_4 \end{Bmatrix} \begin{Bmatrix} j_3 j_2'' x \\ j_4'' j_4 l_1 \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} j_1 & k_1 & k_1' & j_3'' \\ & s_1 & x & s_3 & l_2 \\ j_1'' & k_2' & k_2 & j_3 \end{Bmatrix}. \qquad \text{(A.4.5)}$$

$$\varphi = \sum_{i=1}^{4} s_i + l_1 + l_2,$$

$$\psi = j_2' - j_2 + j_4'' - j_4 + 2s_1 + 2s_3.$$

$$F \qquad \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 & k \\ l_1 & l_2 & l_3 & l_4 & l_1' & l_2' & l_3' & l_4' \\ k' & j_1' & j_2' & j_3' & j_4' \end{Bmatrix} =$$

$$= \begin{Bmatrix} j_4' & j_3' & j_2' & j_1' & k' \\ l_1 & l_4' & l_3' & l_2' & l_1' & l_4 & l_3 & l_2 \\ k & j_4 & j_3 & j_2 & j_1 \end{Bmatrix} =$$

$$= (-1)^{\varphi} \begin{Bmatrix} j_3 & j_2 & j_1 & j_4 & k \\ l_3' & l_1' & l_4 & l_3 & l_2 & l_1 & l_4' & l_3' \\ k' & j_3' & j_1' & j_4' & j_2' \end{Bmatrix}' =$$

$$= \sum_{x_1 x_2} (x_1)(x_2)(-1)^{\psi} \begin{Bmatrix} l_1 & l_4' j_4' \\ x_1 & l_2 j_1 \end{Bmatrix} \begin{Bmatrix} l_1' l_4 j_2' \\ l_3 x_2 j_2 \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} x_1 j_3' & x_2 \\ l_2 j_1' & l_3 \\ j_4' & k' j_2' \end{Bmatrix} \begin{Bmatrix} j_1 & x_1 & x_2 & j_2 \\ l_4' & j_3' & l_1' & k \\ j_4 & l_3' & l_2' & j_3 \end{Bmatrix}. \qquad \text{(A.4.6)}$$

$$\varphi = j_2 + j_4 + k,$$

$$\psi = j_1 - j_2 + j_3 - j_4 - j_1' - j_4' + l_1 + l_1' - l_2 + l_2' + l_3' + l_4 + l_4' - x_1.$$

$$G \qquad \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ & k & k' & \\ l_1' & l_2' & l_3' & l_4' \\ j_1' & j_2' & j_3' & j_4' \end{Bmatrix} = \begin{Bmatrix} j_2 & j_3 & j_4 & j_1 \\ l_2 & l_3 & l_4 & l_1 \\ & k' & k & \\ l_2' & l_3' & l_4' & l_1' \\ j_2' & j_3' & j_4' & j_1' \end{Bmatrix} =$$

$$= (-1)^{\varphi_1} \begin{Bmatrix} j_1' & j_2' & j_3' & j_4' \\ l_1' & l_2' & l_3' & l_4' \\ & k & k' & \\ l_1 & l_2 & l_3 & l_4 \\ j_1 & j_2 & j_3 & j_4 \end{Bmatrix} = (-1)^{\varphi_2} \begin{Bmatrix} j_4' & j_3' & j_2' & j_1' \\ l_1' & l_4' & l_3' & l_2' \\ & k & k' & \\ l_1 & l_4 & l_3 & l_2 \\ j_4 & j_3 & j_2 & j_1 \end{Bmatrix} =$$

$$= \sum_{x} (x)(-1)^{\psi} \begin{Bmatrix} l_1 l_1' k \\ l_3' l_3 x \end{Bmatrix} \begin{Bmatrix} l_2 l_2' k' \\ l_4' l_4 x \end{Bmatrix} \begin{Bmatrix} j_2 l_2 j_1 \\ j_3 l_4 j_4 \\ l_2 x l_1 \end{Bmatrix} \begin{Bmatrix} j_2' l_2' j_1' \\ j_3' l_4' j_4' \\ l_3' x l_1' \end{Bmatrix}. \qquad \text{(A.4.7)}$$

$$\varphi_1 = 2k + 2k',$$

$$\varphi_2 = \sum_{i=1}^{4} (l_i + l_i'),$$

$$\psi = j_1 + j_3 - j_1' - j_3' + l_1 + l_3 + l_1' + l_3' + k - k'.$$

$$H\left\{\begin{matrix} j_1 & j_2 & j_3 & j'_1 & j'_2 & j'_3 & j''_1 & j''_2 & j''_3 \\ k_1 & k_2 & k_3 & k'_1 & k'_2 & k'_3 & k''_1 & k''_2 & k''_3 \end{matrix}\right\} =$$

$$=\left\{\begin{matrix} j''_3 & j_1 & j_2 & j_3 & j'_1 & j'_2 & j''_3 & j''_1 & j''_2 \\ k''_3 & k_1 & k_2 & k_3 & k'_1 & k'_2 & k''_3 & k''_1 & k''_2 \end{matrix}\right\} =$$

$$=\left\{\begin{matrix} j'_1 & j_3 & j_2 & j_1 & j''_3 & j''_2 & j''_1 & j'_3 & j'_2 \\ k_3 & k_2 & k_1 & k''_3 & k''_2 & k''_1 & k'_3 & k'_2 & k'_1 \end{matrix}\right\} =$$

$$=\sum_x (x_1)(x_2)(-1)^{\phi} \left\{\begin{matrix} k_1 & k'_1 & k''_1 \\ x_1 & j_1 & j_2 \end{matrix}\right\} \left\{\begin{matrix} k_2 & k'_2 & k''_2 \\ j''_1 & x_2 & j'_3 \end{matrix}\right\} \times$$

$$\times \left\{\begin{matrix} x_1 & x_2 & k''_2 \\ j_1 & k''_3 & j''_3 \\ k''_1 & j''_2 & j''_2 \end{matrix}\right\} \left\{\begin{matrix} x_2 & x_1 & k'_1 & j'_2 \\ k''_2 & j_2 & j'_1 & j'_3 \\ k'_2 & k_2 & j_3 & k_3 \end{matrix}\right\}. \qquad \text{(A.4.8)}$$

$$\phi = j_2 + j''_2 + j'_3 - j''_3 + k'_1 + k''_2 + k_3 + 2j'_1 - x_1 + x_2.$$

$$I \qquad \begin{bmatrix} j_1 & j_2 & r_1 & j_3 & j_4 \\ & k_1 & & l_2 & k_2 \\ l_1 & & & & l'_1 \\ & k'_1 & & l'_2 & k'_2 \\ j'_1 & j'_2 & r_2 & j'_3 & j'_4 \end{bmatrix} =$$

$$= \begin{bmatrix} j'_1 & j'_2 & r_1 & j'_3 & j'_4 \\ & k'_1 & & l'_2 & k'_2 \\ l_1 & & & & l'_1 \\ & k_1 & & l_2 & k_2 \\ j_1 & j_2 & r_2 & j_3 & j_4 \end{bmatrix} = \begin{bmatrix} j_2 & j_4 & r_2 & j'_4 & j'_2 \\ & k_2 & & l'_1 & k'_2 \\ l_2 & & & & l'_2 \\ & k_1 & & l_1 & k'_1 \\ j_2 & j_1 & r_1 & j'_1 & j'_2 \end{bmatrix} =$$

$$=\sum_x (x)(-1)^{\phi} \left\{\begin{matrix} k_1 & k'_1 & x \\ l'_2 & l_2 & l_1 \end{matrix}\right\} \left\{\begin{matrix} k_2 & k'_2 & x \\ l'_2 & l_2 & l'_1 \end{matrix}\right\} \times$$

$$\times \left\{\begin{matrix} j_4 & j_1 & j_2 & j_2 & k_2 & k_1 \\ r_2 & & r_1 & & x & \\ j'_4 & j'_1 & j''_2 & j'_2 & k'_2 & k'_1 \end{matrix}\right\}. \qquad \text{(A.4.9)}$$

$$\phi = l_2 + l'_2 - r_1 + r_2 - j_1 - j_2 - j'_2 + j_4 + k_1 - k_2.$$

$$K\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ & k_1 \quad l \quad k_2 & \\ r_1 & s_1 & s_2 & r_2 \\ & k_1' \quad l' \quad k_2' & \\ j_1' & j_2' & j_3' & j_4' \end{Bmatrix} = \begin{Bmatrix} j_4 & j_3 & j_2 & j_1 \\ & k_2 \quad l \quad k_1 & \\ r_2 & s_2 & s_1 & r_1 \\ & k_2' \quad l' \quad k_1' & \\ j_4' & j_3' & j_2' & j_1' \end{Bmatrix} =$$

$$= \begin{Bmatrix} j_1' & j_2' & j_3' & j_4' \\ & k_1' \quad l' \quad k_2' & \\ r_1 & s_1 & s_2 & r_2 \\ & k_1 \quad l \quad k_2 & \\ j_1 & j_2 & j_3 & j_4 \end{Bmatrix} = (-1)^{\varphi} \begin{Bmatrix} j_3 & j_4 & j_1 & j_2 \\ & k_2 \quad l \quad k_1 & \\ r_1 & s_2 & s_1 & r_2 \\ & k_2' \quad l' \quad k_1' & \\ j_3' & j_4' & j_1' & j_2' \end{Bmatrix} =$$

$$= \sum_x (x)(-1)^{\psi} \begin{Bmatrix} j_1 & k_1 & j_2 \\ j_3 & k_2 & j_4 \\ r_1 & x & r_2 \end{Bmatrix} \begin{Bmatrix} j_1' & k_1' & j_2' \\ j_3' & k_2' & j_4' \\ r_1 & x & r_2 \end{Bmatrix} \times \begin{Bmatrix} k_1 & k_2 & x \\ s_2 & s_1 & l \end{Bmatrix} \begin{Bmatrix} k_1' & k_2' & x \\ s_2 & s_1 & l' \end{Bmatrix}. \quad \text{(A.4.10)}$$

$$\varphi = k_1 - k_1' + k_2 - k_2' + \sum_{i=1}^{4} (j_i + j_i'),$$

$$\psi = 2r_1 + 2s_2.$$

$$L\begin{vmatrix} j_1 & k_1 & j_1' \\ & m \quad m' & \\ j_2 & k_2 & j_2' \\ & s_1 \quad s_2 & \\ l_1 & p_1 & l_1' \\ & r \quad r' & \\ l_2 & p_2 & l_2' \end{vmatrix} = \begin{vmatrix} j_1' & k_1 & j_1 \\ & m' \quad m & \\ j_2' & k_2 & j_2 \\ & s_1 \quad s_2 & \\ l_2' & p_2 & l_2 \\ & r' \quad r & \\ l_1' & p_1 & l_1 \end{vmatrix} = \begin{vmatrix} j_2 & k_2 & j_2' \\ & m \quad m' & \\ j_1 & k_1 & j_1' \\ & s_2 \quad s_1 & \\ l_2 & p_2 & l_2' \\ & r \quad r' & \\ l_1 & p_1 & l_1' \end{vmatrix} =$$

$$= \sum_x (x)(-1)^{\psi} \begin{Bmatrix} x & j_1 & j_1' \\ s_1 & l_2' & l_1 \end{Bmatrix} \begin{Bmatrix} x & j_2 & j_2' \\ s_2 & l_1' & l_2 \end{Bmatrix} \begin{Bmatrix} x & j_1 & j_1' \\ k_1 & m' & m \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} x & j_2 & j_2' \\ k_2 & m' & m \end{Bmatrix} \begin{Bmatrix} p_1 & r' & l_1' \\ l_1 & l_2' & a \\ r & p_2 & l_2 \end{Bmatrix}. \quad \text{(A.4.11)}$$

$$\psi = j_1 + j_1' + j_2 + j_2'.$$

$$M \qquad \left\{\begin{array}{ccccc} j_1 & l_1 & s & l_2 & j_2 \\ & k_1 & k_2 & & \\ r & p_1 & & p_2 & m \\ & k_1' & k_2' & & \\ j_1' & l_1' & s' & l_2' & j_2' \end{array}\right\} =$$

$$= \left\{\begin{array}{ccccc} j_2 & l_2 & s & l_1 & j_1 \\ & k_2 & k_1 & & \\ r & p_2 & & p_1 & m \\ & k_2' & k_1' & & \\ j_2' & l_2' & s' & l_1' & j_1' \end{array}\right\} = \left\{\begin{array}{ccccc} j_2' & l_2' & s' & l_1' & j_1' \\ & k_2' & k_1' & & \\ r & p_2 & & p_1 & m \\ & k_2 & k_1 & & \\ j_2 & l_2 & s & l_1 & j_1 \end{array}\right\} =$$

$$= \sum_x (x)(-1)^{\psi} \left\{\begin{array}{ccc} m & k_2' & k_1 \\ x & k_2 & k_1' \end{array}\right\} \left\{\begin{array}{ccc} s & l_2 & k_2 \\ x & k_1 & l_1 \end{array}\right\} \left\{\begin{array}{ccc} s' & l_2' & k_2' \\ x & k_1' & l_1' \end{array}\right\} \times$$

$$\times \begin{bmatrix} j_1 & l_2 & j_1' & l_2' \\ r & p_1 & p_2 & x \\ j_2 & j_2' & l_1 & l_1' \end{bmatrix}. \qquad \text{(A.4.12)}$$

$$\psi = j_1 - j_2 - j_1' + j_2' + k_1 + k_2 + k_1' + k_2'.$$

$$N \qquad \begin{pmatrix} j_1 & k_1 & m & k_1' & j_1' \\ & l_1 & p_1 & l_1' & \\ s_1 & & & & s_2 \\ & l_2 & p_2 & l_2' & \\ j_2 & k_2 & r & k_2' & j_2' \end{pmatrix} =$$

$$= \begin{pmatrix} j_2 & k_2 & m & k_2' & j_2' \\ & l_2 & p_2 & l_2' & \\ s_1 & & & & s_2 \\ & l_1 & p_1 & l_1' & \\ j_1 & k_1 & r & k_1' & j_1' \end{pmatrix} = \begin{pmatrix} j_2' & k_2' & m & k_2 & j_2 \\ & l_2' & p_2 & l_2 & \\ s_2 & & & & s_1 \\ & l_1' & p_1 & l_1 & \\ j_1' & k_1' & r & k_1 & j_1 \end{pmatrix} =$$

$$= \sum_x (x)(-1)^{\psi} \left\{\begin{array}{ccc} l_1 & l_2 & x \\ p_2 & p_1 & s_1 \end{array}\right\} \left\{\begin{array}{ccc} l_1' & l_2' & s_2 \\ p_2 & p_1 & x \end{array}\right\} \times$$

$$\times \left\{\begin{array}{cccccc} j_1' & j_2 & k_1' & k_2 & l_1' & l_2 \\ & r & & m & & x \\ j_2' & j_1 & k_2' & k_1 & l_2' & l_1 \end{array}\right\}. \qquad \text{(A.4.13)}$$

$$\psi = j_1' - j_2 + k_1' + k_2' - l_1' + l_2 - m - r + x.$$

$$P\begin{Bmatrix} j & k_1 & l_1 & l_1' & k_1' & j' \\ & & t & & & \\ k_2 & m_1 & & m_1' & k' & \\ & & u & & & \\ l_2' & m_2' & p' & p & m_2 & l_2 \end{Bmatrix} =$$

$$= (-1)^{\varphi}\begin{Bmatrix} j' & k_2' & l_2 & l_2' & k_2 & j \\ & & t & & & \\ k_1' & m_2 & & m_2' & k_1 & \\ & & u & & & \\ l_1' & m_1' & p & p' & m_1 & l_1 \end{Bmatrix} = \begin{Bmatrix} j' & k_1' & l_1' & l_1 & k_1 & j \\ & & t & & & \\ k_2' & m_1' & & m_1 & k_2 & \\ & & u & & & \\ l_2 & m_2 & p & p' & m_2' & l_2' \end{Bmatrix} =$$

$$= \sum_{x_1 x_2} (x_1)(x_2)(-1)^{\psi} \begin{Bmatrix} p & p' & u \\ x_2 & x_1 & t \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} k_1 & k_2 & j \\ k_2' & k_1' & j' \\ x_1 & x_2 & u \end{Bmatrix} \begin{Bmatrix} l_1 & k_1 & m_1 \\ l_2 & k_2' & m_2 \\ t & x_1 & p \end{Bmatrix} \begin{Bmatrix} l_1' & k_1' & m_1' \\ l_2' & k_2 & m_2' \\ t & x_2 & p' \end{Bmatrix}. \qquad \text{(A.4.14)}$$

$$\varphi = j + j' + k_1 + k_1' + p + p' + k_2 + k_2' + u + 2t,$$

$$\psi = j + j' + k_2 + k_2' + l_2 + l_2' - m_2 - m_2' + p + p'.$$

$$R\begin{Bmatrix} j & k & l & m & n & u \\ & & p & & & \\ r & s & & s' & r' & \\ & & p' & & & \\ t & n' & m' & l' & k' & j' \end{Bmatrix} = \begin{Bmatrix} j' & k' & l' & m' & n' & u \\ & & p' & & & \\ r' & s' & & s & r & \\ & & p & & & \\ t & n & m & l & k & j \end{Bmatrix} =$$

$$= \sum_{x} (x)(-1)^{\psi} \begin{Bmatrix} n' & m' & s \\ a & t & r' \end{Bmatrix} \begin{Bmatrix} j' & k' & r' \\ u & s' & x \\ p' & l' & m' \end{Bmatrix} \begin{Bmatrix} m & n & r & j \\ & s' & t & k & p \\ u & & x & s & l \end{Bmatrix}. \qquad \text{(A.4.15)}$$

$$\psi = m + s' + t + r.$$

$$S\begin{Bmatrix} j & k & l & m & n & & t \\ & r & p & s & s' & p' & r' & \\ u & & & n' & m' & l' & k' & j' \end{Bmatrix} = \begin{Bmatrix} j' & k' & l' & m' & n' & & t \\ & r' & p' & s' & s & p & r & \\ u & & & n & m & l & k & j \end{Bmatrix} =$$

$$= \sum_{x} (x)(-1)^{\psi} \begin{Bmatrix} j & x & p \\ j' & u & p' \end{Bmatrix} \begin{Bmatrix} j & k & r \\ l & x & p \end{Bmatrix} \begin{Bmatrix} j' & k' & r' \\ l' & x & p' \end{Bmatrix} \begin{Bmatrix} s & & n' & t & n & \\ & m' & & r' & r & m \\ s' & & l' & & x & l \end{Bmatrix}. \qquad \text{(A.4.16)}$$

$$\psi = j + j' + k + k' + l + l' + p + p' + 2s + 2s' + u - x.$$

$$T \quad \begin{Bmatrix} k & l & m & n & u & n' & m' & l' & k' \\ j & r & s & p & t & p' & s' & r' & j' \end{Bmatrix} =$$

$$= \begin{Bmatrix} k' & l' & m' & n' & u & n & m & l & k \\ j' & r' & s' & p' & t & p & s & r & j \end{Bmatrix} =$$

$$= \sum_x (x)(-1)^{\psi} \begin{Bmatrix} n & m & p \\ l & x & s \end{Bmatrix} \begin{Bmatrix} n' & m' & p' \\ l' & x & s' \end{Bmatrix} \begin{Bmatrix} n & n' & u \\ s' & s & x \end{Bmatrix} \times$$

$$\begin{Bmatrix} r & j' & t & j \\ k' & p' & p & k \\ r' & l' & x & l \end{Bmatrix}. \qquad \text{(A.4.17)}$$

$$\psi = -j - j' + m + m' + n + n' + s + s' - u.$$

$$V \quad \begin{bmatrix} l & k & j & t & j' & k' & l' \\ s & r & & & r' & s' & \\ p & n & m & u & m' & n' & p' \end{bmatrix} =$$

$$= \begin{bmatrix} l' & k' & j' & t & j & k & l \\ s' & r' & & & r & s & \\ p' & n' & m' & u & m & n & p \end{bmatrix} =$$

$$= \sum_{x_1 x_2} (x_1)(x_2) \begin{Bmatrix} n & p & s \\ x_1 & m & u \end{Bmatrix} \begin{Bmatrix} n' & p' & s' \\ x_2 & m' & u \end{Bmatrix} \begin{Bmatrix} x_1 & p & u \\ p' & x_2 & t \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} j & j' & t \\ k & r' & x_1 \\ r & k' & x_2 \end{Bmatrix} \begin{Bmatrix} l & m & r' \\ x_1 & k & s \end{Bmatrix} \begin{Bmatrix} l' & m' & r \\ x_2 & k' & s' \end{Bmatrix}. \qquad \text{(A.4.18)}$$

Appendix 5

Expressions for the transformation matrices of eigenfunctions of five coupled angular momenta

For matrix elements which are expressed in terms of $6j$- and $9j$-coefficients, we only give their expressions in terms of the product of the simplified matrices. In the remaining cases we give expressions for the matrix elements in terms of $12j$-coefficients of the first and second kind. Expressions are given only for those matrices which were included in Table 23.1, as the remainder may be reduced to these by elementary means (cf. the remark in connection with Table 23.1 on page 79). The notation for products of matrices and coupling schemes was stipulated in section 21.

To shorten the notation the abbreviation $[\ldots]^{\frac{1}{2}}$ is used, in which the dotted line denotes the product of the quantities $(2J+1)$ for all intermediate angular momenta on the right and left sides of the transformation matrix.

Matrices susceptible of expression in terms of the product of two $6j$-coefficients

$$\left((12345)^{A_0}\,|\,(13254)^{A_0}\right) =$$

$$= \left((12345)^{A_0}\,|\,(13245)^{A_0}\,|\,(13254)^{A_0}\right). \qquad \text{(A.5.1)}$$

$$\left((12345)^{A_1}\,|\,(35412)^{A_0}\right) =$$

$$= \left((12345)^{A_1}\,|\,(12354)^{A_1}\,|\,(35412)^{A_0}\right). \qquad \text{(A.5.2)}$$

Matrices susceptible of expression in terms of product of three $6j$-coefficients

$$\left((12345)^{A_0}\,|\,(13452)^{A_0}\right) =$$

$$= \left((12345)^{A_0}\,|\,(13245)^{A_0}\,|\,(13425)^{A_0}\,|\,(13452)^{A_0}\right). \qquad \text{(A.5.3)}$$

$$\left((12345)^{A_0}\,|\,(13524)^{A_0}\right) =$$

$$= \left((12345)^{A_0}\,|\,(13245)^{A_0}\,|\,(13254)^{A_0}\,|\,(13524)^{A_0}\right). \qquad \text{(A.5 4)}$$

$$\left((12345)^{A_0}\,|\,(34152)^{A_0}\right) =$$

$$= \left((12345)^{A_0}\,|\,(12345)^{A_2}\,|\,(34125)^{A_0}\,|\,(34152)^{A_0}\right). \qquad \text{(A.5.5)}$$

$$\left((12345)^{A_0}|(34512)^{A_0}\right)=$$

$$=\left((12345)^{A_0}|(12345)^{A_1}|(12345)^{A_1}|(34512)^{A_0}\right). \qquad \text{(A.5.6)}$$

$$\left((12345)^{A_0}|(45312)^{A_0}\right)=$$

$$=\left((12345)^{A_0}|(45123)^{A_1}|(12453)^{A_1}|(45312)^{A_0}\right). \qquad \text{(A.5.7)}$$

$$\left((12345)^{A_1}|(35142)^{A_1}\right)=$$

$$=\left((12345)^{A_1}|(12354)^{A_1}|(35124)^{A_1}|(35142)^{A_1}\right). \qquad \text{(A.5.8)}$$

$$\left((12345)^{A_1}|(13245)^{A_0}\right)=$$

$$=\left((12345)^{A_1}|(12345)^{A_2}|(12345)^{A_0}|(13245)^{A_0}\right). \qquad \text{(A.5.9)}$$

$$\left((12345)^{A_1}|(13425)^{A_0}\right)=$$

$$=\left((12345)^{A_1}|(12345)^{A_2}|(34125)^{A_0}|(13425)^{A_0}\right). \qquad \text{(A.5.10)}$$

$$\left((12345)^{A_1}|(13452)^{A_0}\right)=$$

$$=\left((12345)^{A_1}|(34512)^{A_0}|(34152)^{A_0}|(13452)^{A_0}\right). \qquad \text{(A.5 11)}$$

$$\left((12345)^{A_1}|(15234)^{A_0}\right)=$$

$$=\left((12345)^{A_1}|(34125)^{A_1}|(12534)^{A_0}|(15234)^{A_0}\right). \qquad \text{(A.5.12)}$$

$$\left((12345)^{A_1}|(15342)^{A_0}\right)=$$

$$=\left((12345)^{A_1}|(34512)^{A_0}|(15342)^{A_0}|(15342)^{A_0}\right). \qquad \text{(A.5.13)}$$

$$\left((12345)^{A_1}|(35124)^{A_0}\right)=$$

$$=\left((12345)^{A_1}|(12354)^{A_1}|(12354)^{A_2}|(35124)^{A_0}\right). \qquad \text{(A.5.14)}$$

$$\left((12345)^{A_1}|(35142)^{A_0}\right)=$$

$$=\left((12345)^{A_1}|(12354)^{A_1}|(35412)^{A_0}|(35142)^{A_0}\right). \qquad \text{(A.5.15)}$$

$$\left((12345)^{A_2}|(13254)^{A_0}\right) =$$

$$= \left((12345)^{A_2}|(12345)^{A_0}|(12354)^{A_0}|(13254)^{A_0}\right). \qquad \text{(A.5.16)}$$

$$\left((12345)^{A_2}|(13452)^{A_0}\right) =$$

$$= \left((12345)^{A_2}|(34125)^{A_0}|(13425)^{A_0}|(13452)^{A_0}\right). \qquad \text{(A.5.17)}$$

$$\left((12345)^{A_2}|(15234)^{A_0}\right) =$$

$$= \left((12345)^{A_2}|(34125)^{A_1}|(34152)^{A_1}|(15234)^{A_0}\right). \qquad \text{(A.5.18)}$$

$$\left((12345)^{A_1}|(15234)^{A_1}\right) =$$

$$= \left((12345)^{A_2}|(34215)^{A_0}|(15342)^{A_1}|(15234)^{A_1}\right). \qquad \text{(A.5.19)}$$

Matrices susceptible of expression in terms of products of $6j$- and $9j$-coefficients

$$\left((12345)^{A_0}|(13542)^{A_0}\right) =$$

$$= \left((12345)^{A_0}|(13245)^{A_0}|(13542)^{A_0}\right). \qquad \text{(A.5.20)}$$

$$\left((12345)^{A_0}|(14352)^{A_0}\right) =$$

$$= \left((12345)^{A_0}|(14325)^{A_0}|(14352)^{A_0}\right). \qquad \text{(A.5.21)}$$

$$\left((12345)^{A_1}|(35412)^{A_0}\right) =$$

$$= \left((12345)^{A_0}|(12354)^{A_1}|(35412)^{A_0}\right). \qquad \text{(A.5.22)}$$

$$\left((12345)^{A_1}|(13452)^{A_1}\right) =$$

$$= \left((12345)^{A_1}|(12453)^{A_1}|(13452)^{A_1}\right). \qquad \text{(A.5.23)}$$

$$\left((12345)^{A_1}|(13254)^{A_0}\right) =$$

$$= \left((12345)^{A_1}|(12354)^{A_0}|(13254)^{A_0}\right). \qquad \text{(A.5.24)}$$

$$\left((12345)^{A_1}|(13542)^{A_0}\right) =$$

$$= \left((12345)^{A_1}|(34512)^{A_0}|(13542)^{A_0}\right). \qquad \text{(A.5.25)}$$

$$\left((12345)^{A_2}|(15342)^{A_0}\right)=$$

$$=\left((12345)^{A_2}|(15342)^{A_2}|(15342)^{A_0}\right). \qquad \text{(A.5.26)}$$

$$\left((12345)^{A_2}|(13245)^{A_1}\right)=$$

$$=\left((12345)^{A_2}|(13245)^{A_2}|(13245)^{A_1}\right). \qquad \text{(A.5 27)}$$

Matrices susceptible of expression in terms of $12j$-coefficients of the first kind

$$\left((12345)^{A_0}|(14532)^{A_0}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} j_3 & J_{123} & J_{1234} & J \\ J_{12} & j_4 & j_5 & J_{1345} \\ j_2 & j_1 & J_{14} & J_{145} \end{Bmatrix}, \qquad \text{(A.5.28)}$$

$$\varphi=J_{12}+J_{1345}-J_{145}-J_{123}.$$

$$\left((12345)^{A_0}|(15342)^{A_0}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} j_2 & J_{12} & J_{123} & J_{1234} \\ j_1 & j_3 & j_4 & J \\ j_5 & J_{15} & J_{135} & J_{1345} \end{Bmatrix}, \qquad \text{(A.5.29)}$$

$$\varphi=J_{15}-J_{12}-J_{1345}+J_{1234}.$$

$$\left((12345)^{A_0}|(15432)^{A_0}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} j_1 & J_{12} & J_{123} & J_{1234} \\ j_2 & j_3 & j_4 & j_5 \\ J & J_{1345} & J_{145} & J_{15} \end{Bmatrix}, \qquad \text{(A.5 30)}$$

$$\varphi=j_1-j_2-j_3-j_4-j_5+J.$$

$$\left((12345)^{A_0}|(35142)^{A_0}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} j_5 & J_{35} & J_{135} & j_4 \\ j_3 & j_1 & J_{1345} & J_{1234} \\ J_{123} & J_{12} & j_2 & J \end{Bmatrix}, \qquad \text{(A.5.31)}$$

$$\varphi=j_2+j_4+j_5+J_{12}+J_{35}-J_{123}-J_{135}-J.$$

$$\left((12345)^{A_1}|(13245)^{A_1}\right)=[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} J_{345} & J_{34} & j_3 & J_{12} \\ & j_5 & j_4 & j_1 & J \\ J_{245} & J_{24} & j_2 & J_{12} \end{Bmatrix}. \tag{A.5.32}$$

$$\left((12345)^{A_1}|(13254)^{A_1}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} J_{25} & j_2 & j_1 & j_3 \\ & j_5 & J_{12} & J_{13} & j_4 \\ J_{34} & J_{345} & J & J_{245} \end{Bmatrix}, \tag{A.5.33}$$

$$\varphi=j_1+j_2+j_3+j_4+j_5+J_{12}+J_{13}+J_{345}+J_{245}+J.$$

$$\left((12345)^{A_1}|(13524)^{A_0}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} J_{12} & J_{345} & j_5 & J_{135} \\ & J & J_{34} & J_{12} & j_2 \\ J_{1235} & j_4 & j_3 & j_1 \end{Bmatrix}, \tag{A.5.34}$$

$$\varphi=j_4-j_5+J_{12}+J_{13}-J_{135}-J_{345}-J_{1235}.$$

$$\left((12345)^{A_2}|(13524)^{A_0}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} J_{1235} & j_2 & J_{12} & J_{1234} \\ & J_{135} & j_1 & J_{34} & J \\ j_5 & & J_{13} & j_3 & j_4 \end{Bmatrix}, \tag{A.5.35}$$

$$\varphi=j_1-j_2+j_3-j_4+j_5-J_{12}-J_{34}-J_{1234}-J.$$

$$\left((12345)^{A_2}|(13542)^{A_0}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} J_{1345} & j_4 & J_{34} & J_{1234} \\ & J_{135} & j_3 & J_{12} & J \\ j_5 & & J_{13} & j_1 & j_2 \end{Bmatrix}, \tag{A.5.36}$$

$$\varphi=j_2+j_4+j_5-J_{12}-J.$$

$$\left((12345)^{A_2}|(15324)^{A_0}\right)=(-1)^{\varphi}[\ldots]^{\frac{1}{2}}\times$$

$$\times\begin{Bmatrix} J_{15} & j_5 & J & J_{1235} \\ & j_1 & J_{1234} & j_4 & J_{135} \\ j_2 & & J_{12} & J_{34} & j_3 \end{Bmatrix}, \tag{A.5.37}$$

$$\varphi = j_1 - j_2 - j_3 - j_5 + J_{1235}.$$

$$\left((12345)^{A_1} | (13254)^{A_1}\right) = (-1)^{\varphi} [\ldots]^{\frac{1}{2}} \times$$

$$\times \begin{Bmatrix} J & J_{12} & j_1 & j_2 \\ J_{345} & j_3 & J_{12} & j_5 \\ J_{25} & j_4 & J_{34} & J_{1234} \end{Bmatrix}, \qquad \text{(A.5.38)}$$

$$\varphi = j_2 - J_{25} - J_{1234} + J.$$

$$\left((12345)^{A_1} | (13452)^{A_1}\right) = (-1)^{\varphi} [\ldots]^{\frac{1}{2}} \times$$

$$\times \begin{Bmatrix} J & J_{12} & j_3 & j_4 \\ J_{345} & j_1 & J_{34} & j_5 \\ J_{45} & j_2 & J_{12} & J_{1234} \end{Bmatrix}, \qquad \text{(A.5.39)}$$

$$\varphi = j_1 + j_3 - j_4 - J_{13} + J_{34} + J_{45} + J_{12} - J.$$

Matrices susceptible of expression in terms of $12j$-coefficients of the second kind.

$$\left((12345)^{A_0} | (14523)^{A_1}\right) = (-1)^{\varphi} [\ldots]^{\frac{1}{2}} \times \begin{bmatrix} j_1 & J_{145} & J_{123} & J \\ j_4 & J_{12} & j_5 & J_{1245} \\ J_{14} & J_{1234} & j_2 & j_3 \end{bmatrix}, \qquad \text{(A.5.40)}$$

$$\varphi = j_1 - J_{12} - J_{14} + J_{1234} + J_{1245} - J.$$

$$\left((12345)^{A_1} | (13254)^{A_1}\right) = (-1)^{\varphi} [\ldots]^{\frac{1}{2}} \times \begin{bmatrix} J & J_{25} & J_{34} & j_1 \\ J_{1234} & j_4 & j_2 & J_{13} \\ j_5 & J_{12} & J_{1235} & j_3 \end{bmatrix}, \qquad \text{(A.5.41)}$$

$$\varphi = j_1 - J_{25} - J_{34} + J.$$

$$\left((12345)^{A_1} | (13452)^{A_1}\right) = (-1)^{\varphi} [\ldots]^{\frac{1}{2}} \times \begin{bmatrix} J & J_{12} & J_{45} & j_3 \\ j_5 & J_{1345} & J_{34} & j_1 \\ J_{1234} & j_4 & j_2 & J_{13} \end{bmatrix}, \qquad \text{(A.5.42)}$$

$$\varphi = j_1 - J_{13} - J_{34} + J_{45} + J_{1234} - J.$$

$$\left((12345)^{A_1} | (15324)^{A_1}\right) = (-1)^{\varphi} [\ldots]^{\frac{1}{2}} \times \begin{bmatrix} J_{135} & j_1 & j_4 & J_{345} \\ j_3 & J_{1235} & j_5 & J_{12} \\ J_{15} & J_{34} & j_2 & J \end{bmatrix}, \qquad \text{(A.5.43)}$$

$$\varphi = j_1 - j_5 - J_{15}.$$

Appendix 6

Sum rules on j-coefficients

In this appendix we give the most frequently used sum rules. We have for convenience repeated the sum rules for $6j$-, $9j$- and $12j$-coefficients which appeared in the text. For the sum rules on $3nj$-coefficients, which were given in the text, we have simply indicated the numbers of these formulas. Wherever possible the formulas have been placed in such a way that the succession of formulas of the same type for $6j$-, $9j$- and $12j$-coefficients ends with the general formula. In most cases we confined outselves to formulas containing only one summation parameter.

Orthogonality relations

$$\sum_{x}(x)\begin{Bmatrix} j_1 & j_2 & x \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & x \\ l_1 & l_2 & l_3' \end{Bmatrix} =$$

$$= \delta(l_3, l_3')(l_3)^{-1}\{j_1 l_2 l_3\}\{l_1 j_2 l_3\}. \tag{A. 6.1}$$

$$\sum_{x_1 x_2}(x_1)(x_2)\begin{Bmatrix} x_1 & x_2 & j_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{Bmatrix} x_1 & x_2 & j_3 \\ k_1 & k_2 & k_3' \\ l_1 & l_2 & l_3' \end{Bmatrix} =$$

$$= \delta(k_3, k_3')\,\delta(l_3, l_3')(k_3)^{-1}(l_3)^{-1}\{k_1 k_2 k_3\}\times \tag{A. 6.2}$$

$$\times\{l_1 l_2 l_3\}\{j_3 k_3 l_3\}.$$

$$\sum_{x_1 x_2 x_3}(x_1)(x_2)(x_3)\begin{Bmatrix} j_1 & j_2 & x_1 & x_2 \\ l_1 & l_2 & l_3 & x_3 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix}\times$$

$$\times\begin{Bmatrix} j_1 & j_2 & x_1 & x_2 \\ l_1' & l_2 & l_3 & x_3 \\ k_1 & k_2' & k_3' & k_4 \end{Bmatrix} = \delta(l_1, l_1')\,\delta(k_2, k_2')\,\delta(k_3, k_3')\times$$

$$\times(l_1)^{-1}(k_2)^{-1}(k_3)^{-1}\{j_1 l_1 j_2\}\{k_1 l_1 k_2\}\times$$

$$\times\{k_2 l_2 k_3\}\{k_3 l_3 k_4\}. \tag{A. 6.3}$$

$$\sum_{x_1 x_2 x_3}(x_1)(x_2)(x_3)\begin{bmatrix} x_1 & j_2 & j_3 & j_4 \\ l_1 & x_2 & l_3 & l_4 \\ x_3 & k_2 & k_3 & k_4 \end{bmatrix}\begin{bmatrix} x_1 & j_2 & j_3 & j_4' \\ l_1 & x_2 & l_3 & l_4' \\ x_3 & k_2' & k_3 & k_4 \end{bmatrix} =$$

$$= \delta(j_4, j_4')\,\delta(l_4, l_4')\,\delta(k_2, k_2')(j_4)^{-1}(l_4)^{-1}(k_2)^{-1}\times$$

$$\times\{l_1 k_2 j_3\}\{k_2 l_3 j_4\}\{j_4 l_4 k_4\}\{l_4 k_3 j_2\}. \tag{A. 6.4}$$

$$\sum_{x_1 \dots x_{n-1}} (x_1) \dots (x_{n-1}) \begin{bmatrix} j_1 & x_2 & x_3 & \dots & x_{n-1} & j_n \\ & x_1 & l_2 & l_3 & \dots & l_{n-1} \quad l_n \\ k_1 & k_2 & k_3 & \dots & k_{n-1} & k_n \end{bmatrix} \times$$

$$\times \begin{bmatrix} j_1 & x_2 & x_3 & \dots & x_{n-1} & j_n \\ & x_1 & l_2 & l_3 & \dots & l_{n-1} \quad l'_n \\ k_1 & k_2 & k'_2 & \dots & k'_{n-1} & k'_n \end{bmatrix} =$$

$$= \delta(k_2, k_2) \dots \delta(k_n, k'_n) \delta(l_n, l'_n) (k_2)^{-1} \dots (k_n)^{-1} (l_n)^{-1} \times$$

$$\times \{k_2 l_2 k_2\} \dots \{k_{n-1} l_{n-1} k_n\} \{k_n l_n k_1\} \{j_n l_n j_1\}. \qquad \text{(A. 6.5)}$$

See also (24.1).

Sum rules of the type in which the sum of the product of two j-coefficients reduces to the same j-coefficient

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & x \end{Bmatrix} \begin{Bmatrix} j_1 & l_1 & j'_3 \\ j_2 & l_2 & x \end{Bmatrix} = (-1)^{-j_3 - j'_3} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & j'_3 \end{Bmatrix}. \qquad \text{(A. 6.6)}$$

$$\sum_{x_1 x_2} (x_1)(x_2)(-1)^{x_2} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ x_1 & x_2 & k_3 \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & j'_3 \\ j_1 & l_2 & l'_3 \\ x_1 & x_2 & k_3 \end{Bmatrix} =$$

$$= (-1)^{j_1 + j_2 + l_1 - l_2 + l_3 + l'_3 - k_3} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_2 & l_1 & l_3 \\ l'_3 & j'_3 & k_3 \end{Bmatrix}. \qquad \text{(A. 6.7)}$$

$$\sum_{x_1 x_2 x_3} (x_1)(x_2)(x_3)(-1)^{x_3} \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ & l_1 & l_2 & l_3 & l_4 \\ x_1 & x_2 & x_3 & k_4 \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} j'_1 & j'_2 & j'_3 & j_4 \\ & l_1 & l_2 & k_4 & l_4 \\ x_1 & x_2 & x_3 & l_3 \end{Bmatrix} = (-1)^{-l_1 + l_2 + j_3 + l_4 + j'_3 - j_3 + l_3 - k_4} \times$$

$$\times \begin{Bmatrix} j_1 & j_2 & j_3 & l_3 \\ & l_1 & l_2 & j_4 & l_4 \\ j'_1 & j'_2 & j'_3 & k_4 \end{Bmatrix}. \qquad \text{(A. 6.8)}$$

$$\sum_{x_1 x_2 x_3} (x_1)(x_2)(x_3) \begin{bmatrix} j_1 & j_2 & x_1 & x_2 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & x_3 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ x_3 & k_2 & k_4 & k'_4 \end{bmatrix} =$$

$$= (k_4) \begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k'_4 \end{bmatrix}. \qquad \text{(A. 6.9)}$$

See also (24.6).

Formulas for the expansion of j-coefficients

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{Bmatrix} = \sum_x (x)(-1)^{2x} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & k_3 & x \end{Bmatrix} \times \begin{Bmatrix} l_1 & l_2 & l_3 \\ j_2 & x & k_2 \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & k_3 \\ x & j_1 & l_1 \end{Bmatrix}. \quad \text{(A. 6.10)}$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ & l_1 & l_2 & l_3 & l_4 \\ k_1 & & k_2 & k_3 & k_4 \end{Bmatrix} =$$

$$= (-1)^{j_1+j_2+j_3+j_4+l_1+l_2+l_3+l_4+k_1+k_2+k_3+k_4} \times$$

$$\times \sum_x (x)(-1)^{-x} \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{Bmatrix} \begin{Bmatrix} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} j_4 & k_4 & x \\ j_1 & k_1 & l_4 \end{Bmatrix}. \quad \text{(A. 6.11)}$$

$$\begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} = (-1)^{l_1-l_2-l_3+l_4} \sum_x (x) \begin{Bmatrix} k_1 & k_2 & x \\ j_3 & j_1 & l_1 \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} k_3 & k_4 & x \\ j_3 & j_1 & l_2 \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & x \\ j_4 & j_2 & l_3 \end{Bmatrix} \begin{Bmatrix} k_3 & k_4 & x \\ j_4 & j_2 & l_4 \end{Bmatrix}. \quad \text{(A. 6.12)}$$

See also (17.1) and (17.2).

Recursion formulas

$$\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ & l_1 & l_2 & l_3 & l_4 \\ k_1 & & k_2 & k_3 & k_4 \end{Bmatrix} = (-1)^{j_2+k_2+j_4+k_4} \times$$

$$\times \sum_x (x)(-1)^{2x} \begin{Bmatrix} k_2 & j_4 & x \\ l_2 & j_3 & j_2 \\ k_3 & l_3 & k_4 \end{Bmatrix} \begin{Bmatrix} k_2 & j_4 & x \\ l_4 & l_1 & k_1 \end{Bmatrix} \begin{Bmatrix} k_4 & j_2 & x \\ l_1 & l_4 & j_1 \end{Bmatrix}. \quad \text{(A. 6.13)}$$

$$\begin{bmatrix} j_1 & j_2 & \cdots & j_{n-1} & j_n \\ & l_1 & l_2 & \cdots & l_{n-1} & l_n \\ k_1 & & k_2 & \cdots & k_{n-1} & k_n \end{bmatrix} = (-1)^{j_1-k_1+j_{n-1}-k_{n-1}} \times$$

$$\times \sum_x (x) \begin{bmatrix} j_1 & j_2 & \cdots & j_{n-2} & j_{n-1} \\ & l_1 & l_2 & \cdots & l_{n-2} & x \\ k_1 & & k_2 & \cdots & k_{n-2} & k_{n-1} \end{bmatrix} \times$$

$$\times \begin{Bmatrix} k_1 & l_n & k_n \\ l_{n-1} & k_{n-1} & x \end{Bmatrix} \begin{Bmatrix} j_1 & l_n & j_n \\ l_{n-1} & j_{n-1} & x \end{Bmatrix}. \quad \text{(A. 6.14)}$$

See also (24.8).

Single sum of one $3nj$-coefficient

$$\sum_x (x) \{ j_1 j_2 x \} = (j_1)(j_2). \tag{A. 6.15}$$

$$\sum_x (x)(-1)^x \{ j_1 j_1 x \} = (-1)^{2j_1}(j_1). \tag{A. 6.16}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & x \end{Bmatrix} = (-1)^{2j_2} \{ j_1 j_2 j_3 \}. \tag{A. 6.17}$$

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & j_1 & j_3 \\ l_1 & l_1 & x \end{Bmatrix} = (-1)^{-j_1-l_1} \left[(j_1)(l_1) \right]^{\frac{1}{2}} \times$$

$$\times \delta(j_3, 0). \tag{A. 6.18}$$

$$\sum_x (-1)^x (x) \begin{Bmatrix} j_1 & j_3 & j_2 \\ k_1 & k_3 & k_2 \\ x & k_1 & j_1 \end{Bmatrix} = \left[(j_3) \right]^{-1} (-1)^{2k_1 + j_2 - k_2} \delta(j_3, k_2) \times$$

$$\times \{ j_1 j_2 j_3 \} \{ k_1 j_3 k_3 \}. \tag{A. 6.19}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \\ x & j_1 & k_1 \end{Bmatrix} = (j_3)^{-1} \delta(j_3, k_2) \{ j_1 j_2 j_3 \} \{ k_1 j_3 k_3 \}. \tag{A. 6.20}$$

$$\sum_x (x) \begin{bmatrix} j_1 & j_2 & j_3 & j_2 \\ l_1 & l_2 & l_3 & x \\ k_1 & k_2 & k_3 & k_2 \end{bmatrix} = \delta(j_1, j_3) \delta(k_1, k_2) \times$$

$$\left[(j_1)(k_1) \right]^{-1} \{ j_1 k_3 l_2 \} \{ j_1 k_1 l_1 \} \{ j_2 k_1 l_3 \}. \tag{A. 6.21}$$

$$\sum_x (x)(-1)^x \begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & x \\ k_1 & k_2 & j_4 & j_2 \end{bmatrix} = (-1)^{j_1 + j_2 - j_3 + j_4 - k_1 + k_2} \times$$

$$\times \begin{Bmatrix} j_1 & j_4 & l_2 \\ l_1 & k_2 & j_3 \\ k_1 & l_3 & j_2 \end{Bmatrix}. \tag{A. 6.22}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & j_3 & k_1 \\ & l_1 & l_2 & l_3 & x \\ j_1 & & k_2 & k_3 & k_1 \end{Bmatrix} = \delta(j_2, k_2) \delta(j_3, k_3) \times$$

$$\times \left[(j_2)(j_3) \right]^{-1} \{ j_1 j_2 l_1 \} \{ j_2 j_3 l_2 \} \{ k_1 j_3 l_3 \}. \tag{A. 6.23}$$

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & x \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} = (-1)^{-j_4-k_4} \times$$

$$\times \begin{Bmatrix} l_1 & l_2 & l_3 \\ j_3 & j_1 & j_2 \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ k_3 & k_1 & k_2 \end{Bmatrix}. \qquad \text{(A.6.24)}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & j_3 & x \\ l_1 & l_2 & k_1 & j_3 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} = (-1)^{j_1-k_1+j_3-k_3} \times$$

$$\times \begin{Bmatrix} l_2 & k_4 & l_1 \\ j_1 & j_2 & j_3 \end{Bmatrix} \begin{Bmatrix} l_2 & k_4 & l_1 \\ k_1 & k_2 & k_3 \end{Bmatrix}. \qquad \text{(A. 6.25)}$$

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & j_2 & j_3 & x \\ l_1 & l_2 & l_3 & l_3 \\ j_3 & k_2 & k_3 & k_4 \end{Bmatrix} = (-1)^{2l_3-j_3-k_3-k_4} \times$$

$$\times \delta(k_3, j_3)(j_3)^{-1} \{ j_1 k_4 l_3 \} \begin{Bmatrix} j_1 & j_2 & l_1 \\ j_3 & k_2 & l_2 \end{Bmatrix}. \qquad \text{(A. 6.26)}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & \cdots & j_n & k_1 \\ l_1 & l_2 & \cdots & l_n & x \\ j_1 & k_2 & \cdots & k_n & k_1 \end{Bmatrix} = \delta(j_2, k_2) \cdots \delta(j_n, k_n) \left[(j_2) \cdots (j_n) \right]^{-1} \times$$

$$\times \{ j_1 l_1 j_2 \} \{ j_2 l_2 j_3 \} \cdots \{ j_n l_n k_1 \}. \qquad \text{(A. 6.27)}$$

$$\sum_x (x) \begin{bmatrix} j_1 & j_2 & \cdots & j_n & k_1 \\ l_1 & l_2 & \cdots & l_n & x \\ j_1 & k_2 & \cdots & k_n & k_1 \end{bmatrix} =$$

$$= \delta(j_2, k_2) \cdots \delta(j_n, k_n) \left[(j_2) \cdots (j_n) \right]^{-1} \times$$

$$\times \{ j_1 l_1 j_2 \} \{ j_2 l_2 j_3 \} \cdots \{ j_n l_n k_1 \}. \qquad \text{(A. 6.28)}$$

$$\sum_x (x)(-1)^x \begin{bmatrix} j_1 & j_2 & \cdots & j_n & k_1 \\ l_1 & l_2 & \cdots & l_n & x \\ k_1 & k_2 & \cdots & k_n & j_1 \end{bmatrix} =$$

$$= (-1)^{-j_1-k_1} \begin{Bmatrix} j_1 & j_2 & \cdots & j_n \\ l_1 & l_2 & \cdots & l_n \\ k_1 & k_2 & \cdots & k_n \end{Bmatrix}. \qquad \text{(A. 6.29)}$$

See also (24.11).

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & \cdots & j_{n-1} & x \\ & l_1 & l_2 \cdots & k_1 & j_{n-1} \\ k_1 & k_2 & \cdots & k_{n-1} & k_n \end{Bmatrix} =$$

$$= (-1)^{j_1-k_1-j_{n-1}+k_{n-1}} \begin{bmatrix} j_1 & \cdots & j_{n-2} & j_{n-1} \\ & l_1 \cdots & l_{n-2} & k_n \\ k_1 & \cdots & k_{n-2} & k_{n-1} \end{bmatrix}. \qquad \text{(A. 6.30)}$$

$$\sum_x (x) \begin{bmatrix} j_1 & j_2 & \cdots & j_{n-1} & x \\ & l_1 & l_2 \cdots & j_1 & j_{n-1} \\ k_1 & k_2 & \cdots & k_{n-1} & k_n \end{bmatrix} =$$

$$= (-1)^{j_1-k_1-j_{n-1}+k_{n-1}} \begin{Bmatrix} j_1 & \cdots & j_{n-2} & j_{n-1} \\ & l_1 \cdots & l_{n-2} & k_n \\ k_1 & \cdots & k_{n-2} & k_{n-1} \end{Bmatrix}. \qquad \text{(A. 6.31)}$$

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & j_2 & \cdots & j_{n-1} & x \\ & l_1 & l_2 \cdots & l_{n-1} & l_{n-1} \\ j_{n-1} & k_2 & \cdots & k_{n-1} & k_n \end{Bmatrix} =$$

$$= \delta(k_{n-1}, j_1)(j_1)^{-1} \cdot (-1)^{k_n - j_1 + j_{n-1}} \{ j_1 k_n l_{n-1} \} \times$$

$$\times \begin{Bmatrix} j_1 & j_2 & \cdots & j_{n-2} \\ & l_1 & l_2 \cdots & l_{n-2} \\ j_{n-1} & k_2 & \cdots & k_{n-2} \end{Bmatrix}. \qquad \text{(A. 6.32)}$$

$$\sum_x (x)(-1)^x \begin{bmatrix} j_1 & j_2 & \cdots & j_{n-2} & j_1 & x \\ & l_1 & l_2 \cdots & l_{n-2} & l_{n-1} & l_{n-1} \\ k_1 & k_2 & \cdots & k_{n-2} & k_{n-1} & k_n \end{bmatrix} =$$

$$= \delta(k_1, k_{n-1})(k_1)^{-1} (-1)^{j_1 - k_1 + k_n} \{ k_1 k_n l_{n-1} \} \times$$

$$\times \begin{bmatrix} j_1 & j_2 & \cdots & j_{n-2} \\ & l_1 & l_2 \cdots & l_{n-2} \\ k_1 & k_2 & \cdots & k_{n-2} \end{bmatrix}. \qquad \text{(A. 6.33)}$$

A single sum of products of one $3nj$-coefficient and one $6j$-coefficient

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ x & k_2 & k_3 \end{Bmatrix} \begin{Bmatrix} x & k_2 & k_3 \\ k_1 & j_1 & l_1 \end{Bmatrix} =$$

$$= (-1)^{2k_1} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & k_3 & k_1 \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ j_2 & k_1 & k_2 \end{Bmatrix}. \qquad \text{(A. 6.34)}$$

$$\sum_x (x) \begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & x \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{Bmatrix} j_4 & k_4 & x \\ j_2 & k_3 & l_4 \end{Bmatrix} =$$

$$= (-1)^{j_1+j_2-j_3+j_4-k_1+k_2+k_3+k_4} \begin{Bmatrix} k_3 & j_1 & k_1 & j_2 \\ l_2 & l_1 & l_3 & l_4 \\ k_4 & j_3 & k_2 & j_4 \end{Bmatrix}. \qquad \text{(A. 6.35)}$$

$$\sum_x (x)(-1)^{-x} \begin{bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & x \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{Bmatrix} j_4 & k_4 & x \\ k_3 & j_2 & l_4 \end{Bmatrix} =$$

$$= (-1)^{k_1-k_2-l_3} \begin{Bmatrix} k_1 & k_2 & l_4 \\ j_4 & j_2 & l_3 \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & l_4 \\ j_3 & j_1 & l_1 \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} k_3 & k_4 & l_4 \\ j_3 & j_1 & l_2 \end{Bmatrix}. \qquad \text{(A. 6.36)}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & x \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} \begin{Bmatrix} k_1 & j_4 & x \\ j_1 & k_4 & l_4 \end{Bmatrix} =$$

$$= (-1)^{j_1-j_2+j_3+j_4+k_1+k_2-k_3+k_4} \begin{bmatrix} j_2 & j_4 & k_2 & k_4 \\ l_2 & l_1 & l_3 & l_4 \\ j_3 & k_3 & j_1 & k_1 \end{bmatrix}. \qquad \text{(A. 6.37)}$$

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & x \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} \begin{Bmatrix} k_1 & j_4 & x \\ k_2 & j_1 & l_4 \end{Bmatrix} =$$

$$= (-1)^{j_1+j_2+j_3-j_4-k_1+k_2+k_3+k_4+l_1+l_2+l_3-l_4} \times$$

$$\times \begin{Bmatrix} j_1 & j_2 & l_1 \\ k_2 & k_1 & l_4 \end{Bmatrix} \begin{Bmatrix} j_2 & j_3 & l_2 \\ k_3 & k_2 & l_4 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & l_3 \\ k_4 & k_3 & l_4 \end{Bmatrix}. \qquad \text{(A. 6.38)}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ x & k_2 & k_3 & k_4 \end{Bmatrix} \begin{Bmatrix} l_1 & k_2 & x \\ j_4 & l_4 & k_1 \end{Bmatrix} =$$

$$= (-1)^{j_2+l_2+j_3+k_3-k_1} \begin{Bmatrix} j_2 & k_3 & k_1 \\ j_3 & l_3 & j_4 \\ l_2 & k_3 & k_2 \end{Bmatrix} \begin{Bmatrix} j_2 & k_4 & k_1 \\ l_4 & l_1 & j_1 \end{Bmatrix}. \qquad \text{(A. 6.39)}$$

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ & l_1 & l_2 & l_3 & l_4 \\ x & & k_2 & k_3 & k_4 \end{Bmatrix} \begin{Bmatrix} l_1 & k_2 & x \\ l_4 & j_4 & k_1 \end{Bmatrix} =$$

$$= (-1)^{j_2+j_3+j_4+l_2+l_3+k_2+k_3+k_4-k_1} \times \begin{Bmatrix} j_1 & j_2 & l_2 & k_3 \\ & l_1 & j_3 & k_2 & l_4 \\ k_1 & & j_4 & l_3 & k_4 \end{Bmatrix}. \qquad \text{(A. 6.40)}$$

$$\sum_x (x)(-1)^{2x} \begin{Bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} & x \\ k_1 & & \cdots & k_{n-1} & k_n \end{Bmatrix} \begin{Bmatrix} j_1 & k_n & x \\ k_1 & j_n & l_n \end{Bmatrix} =$$

$$= \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} & l_n \\ k_1 & & \cdots & k_{n-1} & k_n \end{bmatrix}. \qquad \text{(A. 6.41)}$$

$$\sum_x (x) \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} & x \\ k_1 & & \cdots & k_{n-1} & k_n \end{bmatrix} \begin{Bmatrix} k_1 & k_n & x \\ j_1 & j_n & l_n \end{Bmatrix} =$$

$$= (-1)^{2l_n} \begin{Bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} & l_n \\ k_1 & & \cdots & k_{n-1} & k_n \end{Bmatrix}. \qquad \text{(A. 6.42)}$$

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} & x \\ k_1 & & \cdots & k_{n-1} & k_n \end{Bmatrix} \begin{Bmatrix} j_1 & k_n & x \\ j_n & k_1 & l_n \end{Bmatrix} =$$

$$= (-1)^{R_{n-1}+(n-1)l_n-j_n+k_n+2k_1} \times$$

$$\times \begin{Bmatrix} j_1 & j_2 & l_1 \\ k_2 & k_1 & l_n \end{Bmatrix} \begin{Bmatrix} j_2 & j_3 & l_2 \\ k_3 & k_2 & l_n \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-2} & j_{n-1} & l_{n-2} \\ k_{n-1} & k_{n-2} & l_n \end{Bmatrix} \times \begin{Bmatrix} j_{n-1} & j_n & l_{n-1} \\ k_n & k_{n-1} & l_n \end{Bmatrix}. \qquad \text{(A. 6.43)}$$

$$\sum_x (x)(-1)^x \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} & x \\ k_1 & & \cdots & k_{n-1} & k_n \end{bmatrix} \begin{Bmatrix} j_1 & j_n & x \\ k_n & k_1 & l_n \end{Bmatrix} =$$

$$= (-1)^{R_{n-1}+nl_n-j_n+k_n+2j_1} \begin{Bmatrix} j_1 & j_2 & l_1 \\ k_2 & k_1 & l_n \end{Bmatrix} \begin{Bmatrix} j_2 & j_3 & l_2 \\ k_3 & k_2 & l_n \end{Bmatrix} \cdots$$

$$\cdots \begin{Bmatrix} j_{n-2} & j_{n-1} & l_{n-2} \\ k_{n-1} & k_{n-2} & l_n \end{Bmatrix} \begin{Bmatrix} j_{n-1} & j_n & l_{n-1} \\ k_n & k_{n-1} & l_n \end{Bmatrix}. \qquad \text{(A. 6.44)}$$

$$R_{n-1}=\sum_{i=1}^{n-1}(j_i+l_i+k_i)$$

$$\sum_x (x)\begin{Bmatrix} j_1 & \cdots & j_{n-1} & & x \\ & l_1 & \cdots & l_{n-1} & l_n \\ k_1 & & \cdots & k_{n-1} & k_n \end{Bmatrix}\begin{Bmatrix} k_1 & l_n & x \\ l_{n-1} & j_{n-1} & j_n \end{Bmatrix}=$$

$$=(-1)^{j_1-k_1-j_{n-1}+k_{n-1}}\begin{Bmatrix} j_1 & \cdots & j_{n-2} & j_{n-1} \\ & l_1 & \cdots & l_{n-2} & j_n \\ k_1 & & \cdots & k_{n-2} & k_{n-1} \end{Bmatrix}\times$$

$$\begin{Bmatrix} j_1 & l_n & k_n \\ l_{n-1} & k_{n-1} & j_n \end{Bmatrix}. \qquad \text{(A. 6.45)}$$

$$\sum_x (x)\begin{bmatrix} j_1 & \cdots & j_{n-1} & & x \\ & l_1 & \cdots & l_{n-1} & l_n \\ k_1 & & \cdots & k_{n-1} & k_n \end{bmatrix}\begin{Bmatrix} j_1 & l_n & x \\ l_{n-1} & j_{n-1} & j_n \end{Bmatrix}=$$

$$=(-1)^{j_1-k_1+j_{n-1}-k_{n-1}}\begin{bmatrix} j_1 & \cdots & j_{n-2} & j_{n-1} \\ & l_1 & \cdots & l_{n-2} & j_n \\ k_1 & & \cdots & k_{n-2} & k_{n-1} \end{bmatrix}\times$$

$$\times\begin{Bmatrix} k_1 & l_n & k_n \\ l_{n-1} & k_{n-1} & j_n \end{Bmatrix}. \qquad \text{(A. 6.46)}$$

A single sum of products of one $3nj$-coefficient and $9j$- and more complicated coefficients

$$\sum_x (x)\begin{Bmatrix} j_1 & j_2 & x \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & x \\ k_1' & k_2' & k_3 \\ l_1' & l_2' & l_3 \end{Bmatrix}=(-1)^{k_2-l_1-k_1'+l_1'}\begin{bmatrix} k_2 & l_1 & k_2' & l_1' \\ j_2 & k_3 & l_3 & j_1 \\ l_2 & l_2' & k_1 & k_1' \end{bmatrix}. \qquad \text{(A. 6.47)}$$

$$\sum_x (x)(-1)^x\begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 & \cdots & l_{n-1} & x \\ k_1 & & \cdots & k_{n-1} & k_n \end{bmatrix}\times$$

$$\times\begin{Bmatrix} k_1 & k_n & x \\ j_{n+1} & l_n & j_n \\ l_{n+1} & k_{n+1} & j_1 \end{Bmatrix}=(-1)^{-j_{n+1}-k_{n+1}+l_n+l_{n+1}}\times$$

$$\times\begin{Bmatrix} j_1 & \cdots & j_{n-1} & j_n & j_{n+1} \\ & l_1 & \cdots & l_{n-1} & l_n & l_{n+1} \\ k_1 & & \cdots & k_{n-1} & k_n & k_{n+1} \end{Bmatrix}. \qquad \text{(A. 6.48)}$$

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & \cdots & j_{n-1} & j_n \\ & l_1 \cdots & & l_{n-1} \quad x \\ k_1 & \cdots & k_{n-1} & k_n \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} j_1 & k_n & x \\ j_{n+1} & l_n & j_n \\ l_{n+1} & k_{n+1} & k_1 \end{Bmatrix} = (-1)^{j_{n+1}+k_{n+1}+l_n+l_{n+1}} \times$$

$$\times \begin{bmatrix} j_1 & \cdots & j_{n-1} & j_n & j_{n+1} \\ & l_1 \cdots & & l_{n-1} \quad l_n & l_{n+1} \\ k_1 & \cdots & k_{n-1} & k_n & k_{n+1} \end{bmatrix}. \qquad \text{(A. 6.49)}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & \cdots & j_{m-1} & j_m \\ & l_1 \cdots & & l_{m-1} \quad x \\ k_1 & \cdots & k_{m-1} & k_m \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} k_1 & j_m & \cdots & j_n \\ x & l_m & \cdots & l_n \\ j_1 & k_m & \cdots & k_n \end{Bmatrix} = (-1)^{j_1-k_1-j_m+k_m} \times$$

$$\times \begin{bmatrix} j_1 & \cdots & j_{m-1} & j_m & \cdots & j_n \\ & l_1 \cdots & & l_{m-1} \quad l_m & \cdots & l_n \\ k_1 & \cdots & k_{m-1} & k_m & \cdots & k_n \end{bmatrix}. \qquad \text{(A. 6.50)}$$

$$\sum_x (x) \begin{bmatrix} j_1 & \cdots & j_{m-1} & j_m \\ & l_1 \cdots & & l_{m-1} \quad x \\ k_1 & \cdots & k_{m-1} & k_m \end{bmatrix} \times$$

$$\times \begin{bmatrix} j_1 & j_m & \cdots & j_n \\ x & l_m & \cdots & l_n \\ k_1 & k_m & \cdots & k_n \end{bmatrix} = (-1)^{j_1-k_1+j_m-k_m} \times$$

$$\times \begin{bmatrix} j_1 & \cdots & j_{m-1} & j_m & \cdots & j_n \\ & l_1 \cdots & & l_{m-1} \quad l_m & \cdots & l_n \\ k_1 & \cdots & k_{m-1} & k_m & \cdots & k_n \end{bmatrix}. \qquad \text{(A. 6.51)}$$

$$\sum_x (x) \begin{Bmatrix} j_1 & \cdots & j_{m-1} & j_m \\ & l_1 \cdots & & l_{m-1} \quad x \\ k_1 & \cdots & k_{m-1} & k_m \end{Bmatrix} \times \begin{bmatrix} k_1 & j_m & \cdots & j_n \\ x & l_m & \cdots & l_n \\ j_1 & k_m & \cdots & k_n \end{bmatrix} = (-1)^{j_1-k_1+j_m-k_m} \times$$

$$\times \begin{Bmatrix} j_1 & \cdots & j_{m-1} & j_m & \cdots & j_n \\ & l_1 \cdots & & l_{m-1} \quad l_m & \cdots & l_n \\ k_1 & \cdots & k_{m-1} & k_m & \cdots & k_n \end{Bmatrix}. \qquad \text{(A. 6.52)}$$

A few other sum rules

$$\sum_x (x)(-1)^x \begin{Bmatrix} j_1 & x & l_1 \\ l_2 & k_3 & j_2 \end{Bmatrix} \begin{Bmatrix} j_2 & x & l_2 \\ l_3 & k_1 & j_3 \end{Bmatrix} \begin{Bmatrix} j_3 & x & l_3 \\ l_1 & k_2 & j_1 \end{Bmatrix} =$$

$$= (-1)^{-(j_1+j_2+j_3+k_1+k_2+k_3+l_1+l_2+l_3)}$$

$$\begin{Bmatrix} k_1 & k_2 & k_3 \\ j_1 & j_2 & j_3 \end{Bmatrix} \begin{Bmatrix} k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}. \qquad \text{(A. 6.53)}$$

$$\sum_{x_1 x_2} (-1)^{x_1} (x_1)(x_2) \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ & l_1 & l_2 & l_3 & l_4 \\ k_1 & & k_2 & x_1 & x_2 \end{Bmatrix} \begin{Bmatrix} l_4 & l_2 & p_1 \\ x_2 & x_1 & l_3 \\ j_1 & k_2 & p_2 \end{Bmatrix} =$$

$$= (-1)^{j_4+2j_1+j_2+k_2} \begin{Bmatrix} j_2 & l_2 & j_3 \\ p_2 & p_1 & l_3 \\ k_1 & l_4 & j_4 \end{Bmatrix} \begin{Bmatrix} j_2 & k_1 & p_2 \\ k_2 & j_1 & l_1 \end{Bmatrix}. \qquad \text{(A. 6.54)}$$

See also (24.10).

$$\sum_{x_2 \cdots x_{k-1}} (x_2) \cdots (x_{k-1}) \times$$

$$\times \begin{Bmatrix} k_1 & k_2 & \cdots & k_{k-1} & k_k & \cdots & k_n \\ & l_1 & l_2 & \cdots & l_{k-1} & l_k & \cdots & l_n \\ j_1 & x_2 & \cdots & x_{k-1} & j_k & \cdots & j_n \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} k'_1 & k'_2 & \cdots & k'_{k-1} & k'_k & k'_{k+1} & \cdots & k'_m \\ & l_1 & l_2 & \cdots & l_{k-1} & l'_k & l'_{k+1} & \cdots & l'_m \\ j_1 & x_2 & \cdots & x_{k-1} & j_k & j'_{k+1} & \cdots & j'_m \end{Bmatrix} =$$

$$= (-1)^{k_1 - k'_1 - k_{k-1} + k'_{k-1}} \sum_x (x) \begin{Bmatrix} k_1 & k_k & \cdots & k_n \\ & x & l_k & \cdots & l_n \\ j_1 & j_k & \cdots & j_n \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} k'_1 & k'_k & k'_{k+1} & \cdots & k'_m \\ & x & l'_k & l'_{k+1} & \cdots & l'_m \\ j_1 & j_k & j'_{k+1} & \cdots & j'_m \end{Bmatrix} \times$$

$$\times \begin{Bmatrix} k_1 & k_2 & \cdots & k_{k-1} & k_k \\ & l_1 & l_2 & & l_{k-1} & x \\ k'_1 & k'_2 & \cdots & k'_{k-1} & k'_k \end{Bmatrix}. \qquad \text{(A. 6.55)}$$

Appendix 7

The simplest summation and transformation formulas for jm-coefficients

In this appendix formulas are given only for those jm-coefficients the diagrams of which contain no more than one closed cycle.

$$\sum_{jm}(j)\,(-1)^{j-m}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & \\ -m_1' & -m_2' & -m \end{pmatrix}=$$

$$=\{j_1\,j_2\,j\}(-1)^{j_1-m_1}\,\delta(m_1,\,m_1')(-1)^{j_2-m_2}\,\delta(m_2,\,m_2'). \qquad \text{(A. 7.1)}$$

$$\sum_{m_1m_2}(-1)^{j_1-m_1+j_2-m_2}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j' \\ -m_1 & -m_2 & -m' \end{pmatrix}=$$

$$=\delta(j,\,j')\;(j)^{-1}\;\{j_1\,j_2\,j\}(-1)^{j-m}\,\delta(m,\,m'). \qquad \text{(A. 7.2)}$$

$$\sum_{m}(-1)^{j-m}\begin{pmatrix} j & j & j' \\ m & -m & m' \end{pmatrix}=(j)^{\frac{1}{2}}\,\delta(j',0)\,\delta(m',0). \qquad \text{(A. 7.3)}$$

$$\sum_{n_1n_2n_3}(-1)^{l_1-n_1+l_2-n_2+l_3-n_3}\times\begin{pmatrix} l_1 & j_1 & l_2 \\ n_1 & m_1 & -n_2 \end{pmatrix}\begin{pmatrix} l_2 & j_2 & l_3 \\ n_2 & m_2 & -n_3 \end{pmatrix}\begin{pmatrix} l_3 & j_3 & l_1 \\ n_3 & m_3 & -n_1 \end{pmatrix}=$$

$$=(-1)^{j_1+j_2+j_3}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_3 & l_1 & l_2 \end{Bmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \qquad \text{(A. 7.4)}$$

$$\sum_{n_1n_2n_3n_4}(-1)^{l_1-n_1+l_2-n_2+l_3-n_3+l_4-n_4}\times$$

$$\times\begin{pmatrix} l_1 & j_1 & l_2 \\ n_1 & m_1 & -n_2 \end{pmatrix}\begin{pmatrix} l_2 & j_2 & l_3 \\ n_2 & m_2 & -n_3 \end{pmatrix}\begin{pmatrix} l_3 & j_3 & l_4 \\ n_3 & m_3 & -n_4 \end{pmatrix}\begin{pmatrix} l_4 & j_4 & l_1 \\ n_4 & m_4 & -n_1 \end{pmatrix}=$$

$$=\sum_{a\mu}(a)\,(-1)^{a-\mu}\,.\,(-1)^{l_3-l_4+a}\times$$

$$\times\begin{Bmatrix} l_4 & l_2 & a \\ j_2 & j_3 & l_3 \end{Bmatrix}\begin{Bmatrix} l_4 & l_2 & a \\ j_1 & j_4 & l_1 \end{Bmatrix}\begin{pmatrix} j_1 & j_4 & a \\ m_1 & m_4 & \mu \end{pmatrix}\times\begin{pmatrix} a & j_3 & j_2 \\ -\mu & m_3 & m_2 \end{pmatrix} \qquad \text{(A. 7.5)}$$

$$=\sum_{a\mu}(a)\,(-1)^{a-\mu}\,.\,(-1)^{j_3+j_4+l_2-l_4+a}\begin{Bmatrix} l_1 & l_2 & j_1 \\ j_4 & j_2 & a \\ l_4 & l_3 & j_3 \end{Bmatrix}\times$$

$$\times\begin{pmatrix} j_1 & j_3 & a \\ m_1 & m_3 & \mu \end{pmatrix}\begin{pmatrix} a & j_4 & j_2 \\ -\mu & m_4 & m_2 \end{pmatrix}. \qquad \text{(A. 7.6)}$$

$$\sum_{n_1 n_2 n_3 n_4 n_5} (-1)^{l_1-n_1+l_2-n_2+l_3-n_3+l_4-n_4+l_5-n_5} \times$$

$$\times \begin{pmatrix} l_1 & j_1 & l_2 \\ n_1 & m_1 & -n_2 \end{pmatrix} \begin{pmatrix} l_2 & j_2 & l_3 \\ n_2 & m_2 & -n_3 \end{pmatrix} \begin{pmatrix} l_3 & j_3 & l_4 \\ n_3 & m_3 & -n_4 \end{pmatrix} \times$$

$$\times \begin{pmatrix} l_4 & j_4 & l_5 \\ n_4 & m_4 & -n_5 \end{pmatrix} \begin{pmatrix} l_5 & j_5 & l_1 \\ n_5 & m_5 & -n_1 \end{pmatrix} =$$

$$= \sum_{a_1 a_2 \mu_1 \mu_2} (a_1)(a_2)(-1)^{a_1-\mu_1+a_2-\mu_2} (-1)^{j_1-j_2+j_3+j_4+j_5+l_5+l_6+a_1-a_2} \times$$

$$\times \begin{Bmatrix} j_2 & j_3 & a_1 \\ l_4 & l_2 & l_3 \end{Bmatrix} \begin{Bmatrix} a_1 & j_4 & a_2 \\ l_5 & l_2 & l_4 \end{Bmatrix} \begin{Bmatrix} a_2 & j_5 & j_1 \\ l_1 & l_2 & l_5 \end{Bmatrix} \times$$

$$\times \begin{pmatrix} j_2 & j_3 & a_1 \\ m_2 & m_3 & -\mu_1 \end{pmatrix} \begin{pmatrix} a_1 & j_4 & a_2 \\ \mu_1 & m_4 & \mu_2 \end{pmatrix} \begin{pmatrix} a_2 & j_5 & j_1 \\ -\mu_2 & m_5 & m_1 \end{pmatrix} \qquad \text{(A. 7.7a)}$$

$$= \sum_{a_1 a_2 \mu_1 \mu_2} (a_1)(a_2)(-1)^{a_1-\mu_1+a_2-\mu_2} \times$$

$$\times (-1)^{j_1-j_2+j_3+l_4+l_5+a_1-a_2} \begin{Bmatrix} l_4 & a_1 & l_2 \\ l_5 & j_5 & l_1 \\ j_4 & a_2 & j_1 \end{Bmatrix} \begin{Bmatrix} l_4 & a_1 & l_2 \\ j_2 & l_3 & j_3 \end{Bmatrix} \times$$

$$\times \begin{pmatrix} j_2 & j_3 & a_1 \\ m_2 & m_3 & -\mu_1 \end{pmatrix} \begin{pmatrix} a_1 & j_5 & a_2 \\ \mu_1 & m_5 & \mu_2 \end{pmatrix} \begin{pmatrix} a_2 & j_4 & j_1 \\ -\mu_2 & m_4 & m_1 \end{pmatrix} \qquad \text{(A. 7.7b)}$$

$$= \sum_{a_1 a_2 \mu_1 \mu_2} (a_1)(a_2)(-1)^{a_1-\mu_1+a_2-\mu_2} (-1)^{2j_2+j_4+j_5+l_1+l_4} \times$$

$$\times \begin{Bmatrix} l_1 & l_2 & l_3 & l_4 \\ j_1 & j_2 & j_3 & l_5 \\ j_4 & a_1 & a_2 & j_5 \end{Bmatrix} \times$$

$$\times \begin{pmatrix} j_4 & j_1 & a_1 \\ m_4 & m_1 & \mu_1 \end{pmatrix} \begin{pmatrix} a_1 & j_2 & a_2 \\ -\mu_1 & m_2 & \mu_2 \end{pmatrix} \begin{pmatrix} a_2 & j_3 & j_5 \\ -\mu_2 & m_3 & m_5 \end{pmatrix}. \qquad \text{(A. 7.7c)}$$

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angular momenta,
contracted 32
free 32

blocks, open and closed 39

coefficients,
Clebsch-Gordan 6
generalized Clebsch-Gordan 17
generalized Wigner 27
j- 33
jm- 31, 32
Wigner 23
3*nj*-, first and second kinds 55

coupling,
scheme 17
sequence 17

diagrams,
block 38
branching form 38
closed 36
contraction of 36
deformation of 35
open 36
rotation of 35
separable 42

distribution (of intermediate couplings) 17

double tensors 114

graphical representation of *jm*-coefficients 34, 35

line,
direction of 35
Hamilton 51

lines,
contraction of 36
corresponding 36
free 36

matrix, transformation 21, 22
renumbering of subscripts 72

node, orientation of 35

$j_2 = 4$

k \ m_3	0	1	2	3	4
−4	$\left[70(j_1+m)^{(4)}(j_1-m)^{(4)}\right]^{\frac{1}{8}}$	$-2\left[14(j_1+m-1)^{(3)}(j_1-m+1)^{(3)}\right]^{\frac{1}{8}}$	$2\left[7(j_1+m-2)^{(3)}(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$-2\left[2(j_1+m-3)(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$\left[(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$
−3	$-4m\left[35(j_1+m)^{(3)}(j_1-m)^{(3)}\right]^{\frac{1}{8}}$	$2(j_1+4m-3)\left[7(j_1+m-1)^{(3)}(j_1-m+1)^{(4)}\right]^{\frac{1}{8}}$	$-2(j_1+2m-3)\left[14(j_1+m-2)(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$2(3j_1+4m-9)\left[(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$-2\left[2(j_1+m-3)(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$
−2	$-\left[(j_1+m+1)(2j_1-8m-1)+3(2m-1)(j_1-m+1)\right]\left[10(j_1+m)^{(3)}(j_1-m)^{(3)}\right]^{\frac{1}{8}}$	$\left[(2j_1+8m-9)(j_1-m+2)-10(2m-1)(j_1+m)\right]\left[2(j_1+m-1)(j_1-m+1)^{(3)}\right]^{\frac{1}{8}}$	$\left[(j_1+m-1)(4j_1+26m-33)(-2m-5)(j_1-m+3)\right]\left[(j_1-m+2)^{(4)}\right]^{\frac{1}{8}}$	$-(2j_1+4m-7)\left[14(j_1+m-2)(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$2\left[7(j_1+m-2)^{(3)}(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$
−1	$2\left[(j_1+m+2)^{(3)}(5j_1-7m+3)-(5j_1+7m+3)(j_1-m+2)^{(3)}\right]\left[5(j_1+m)(j_1-m)^{(3)}\right]^{\frac{1}{8}}$	$-\left[20(j_1+m+1)^{(3)}(j_1-2m+2)-2(7j_1+8m-2)(j_1-m+3)^{(3)}\right]\left[(j_1-m+1)^{(3)}\right]^{\frac{1}{8}}$	$2\left[(j_1+4m-6)(j_1-m+3)-10(m-1)(j_1+m)\right]\left[2(j_1+m-1)(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$2(j_1+4m-6)\left[7(j_1+m-1)^{(3)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$-2\left[14(j_1+m-1)^{(3)}(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$
0	$\left[36(j_1+m+2)^{(3)}(j_1-m+2)^{(3)}(-15j_1+17m+12)(j_1-m+3)^{(3)}-(j_1+m+3)^{(3)}(15j_1-17m+12)\right]^{\frac{1}{8}}$	$2\left[(j_1+m+2)^{(3)}(5j_1-7m+9)-(j_1-m+3)^{(3)}(5j_1+7m+2)\right]\left[5(j_1+m)(j_1-m+1)\right]^{\frac{1}{8}}$	$-\left[(2j_1+8m-7)(j_1-m+3)-3(2m-1)(j_1+m+1)\right]\left[10(j_1+m)^{(3)}(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$-2(2m-3)\left[35(j_1+m)^{(3)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$\left[70(j_1+m)^{(4)}(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$
1	$-2\left[(j_1+m+3)^{(3)}(5j_1-7m+8)-(5j_1+7m+8)(j_1-m+3)^{(3)}\right]\left[5j_1+m+1)^{(3)}(j_1-m+1)\right]^{\frac{1}{8}}$	$2\left[10(j_1+2m+1)(j_1-m+3)^{(3)}-(j_1+m+3)^{(3)}(7j_1-8m+13)\right]\left[(j_1+m+1)^{(3)}\right]^{\frac{1}{8}}$	$2\left[(j_1+11m-8)(j_1-m+3)-3m(7j_1+m+2)\right]\left[2(j_1+m+1)^{(3)}(j_1-m+2)\right]^{\frac{1}{8}}$	$-2(j_1-4m+7)\left[7(j_1+m+1)^{(4)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$-2\left[14(j_1+m+1)^{(3)}(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$
2	$-\left[(j_1+m+3)(3j_1-8m+3)+3(2m-1)(j_1-m+3)\right]\left[10(j_1+m+2)^{(3)}(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$-\left[(j_1+m+3)(2j_1-8m+3)+10(2m-1)(j_1-m+3)\right]\left[2(j_1+m+2)^{(3)}(7j_1-m+3)\right]^{\frac{1}{8}}$	$\left[(j_1+m+4)^{(3)}+3(j_1-m+3)(j_1-9m+8)\right]\left[(j_1+m+2)^{(4)}\right]^{\frac{1}{8}}$	$(3j_1-4m+9)\left[14(j_1+m+2)^{(3)}(j_1-m+3)\right]^{\frac{1}{8}}$	$2\left[7(j_1+m+2)^{(3)}(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$
3	$4m\left[35(j_1+m+3)^{(3)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$-2(j_1-4m+4)\left[7(j_1+m+3)^{(4)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$-2(j_1-2m+4)\left[14(j_1+m+3)^{(3)}(j_1-m+3)\right]^{\frac{1}{8}}$	$-2(3j_1-4m+12)\left[(j_1+m+3)^{(3)}\right]^{\frac{1}{8}}$	$-2\left[2(j_1+m+3)^{(3)}(j_1-m+4)\right]^{\frac{1}{8}}$
4	$\left[70(j_1+m+4)^{(4)}(j_1-m+4)^{(4)}\right]^{\frac{1}{8}}$	$2\left[14(j_1+m+4)^{(3)}(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$	$2\left[7(j_1+m+4)^{(3)}(j_1-m+4)^{(3)}\right]^{\frac{1}{8}}$	$2\left[2(j_1+m+4)^{(3)}(j_1-m+4)\right]^{\frac{1}{8}}$	$\left[(j_1+m+4)^{(3)}\right]^{\frac{1}{8}}$

$j_2 = \frac{7}{2}$

k \ m_3	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
$-\frac{7}{2}$	$\left[35(j_1+m-\frac{1}{2})^{(3)}(j_1-m+\frac{1}{2})^{(4)}\right]^{\frac{1}{8}}$	$-\left[21(j_1+m-\frac{3}{2})^{(3)}(j_1-m+\frac{3}{2})^{(3)}\right]^{\frac{1}{8}}$	$\left[7(j_1+m-\frac{5}{2})(j_1-m+\frac{5}{2})^{(3)}\right]^{\frac{1}{8}}$	$-\left[(j_1-m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$
$-\frac{5}{2}$	$-\left(j_1+7m-\frac{5}{2}\right)\left[5(j_1+m-\frac{1}{2})^{(3)}(j_1-m+\frac{1}{2})^{(3)}\right]^{\frac{1}{8}}$	$\left(3j_1+7m-\frac{15}{2}\right)\left[3(j_1+m-\frac{3}{2})(j_1-m+\frac{3}{2})^{(3)}\right]^{\frac{1}{8}}$	$-\left(5j_1+7m-\frac{25}{2}\right)\left[(j_1-m+\frac{5}{2})^{(3)}\right]^{\frac{1}{8}}$	$\left[7(j_1+m-\frac{5}{2})(j_1-m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$
$-\frac{3}{2}$	$-\left[(j_1+3m-\frac{3}{2})(j_1-m+\frac{3}{2})-4m(j_1+m+\frac{1}{2})(j_1-m+\frac{1}{2})^{(3)}\right]^{\frac{1}{8}}$	$-\left[(j_1-m+\frac{7}{2})^{(3)}+20(m-1)(j_1+m-\frac{1}{2})\right]\left[(j_1-m+\frac{3}{2})^{(3)}\right]^{\frac{1}{8}}$	$\left(3j_1+7m-\frac{19}{2}\right)\left[3(j_1+m-\frac{3}{2})(j_1-m+\frac{5}{2})^{(3)}\right]^{\frac{1}{8}}$	$-\left[21(j_1+m-\frac{3}{2})^{(3)}(j_1-m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$
$-\frac{1}{2}$	$\left[(j_1+m+\frac{3}{2})^{(3)}(14j_1-22m+17)-(11j_1+13m+\frac{5}{2})(j_1-m+\frac{1}{2})^{(3)}\right]^{\frac{1}{8}}$	$-\left[(j_1+3m-\frac{5}{2})(j_1-m+\frac{5}{2})-2(2m-1)(j_1+m+\frac{1}{2})\right]\left[15(j_1+m-\frac{1}{2})(j_1-m+\frac{3}{2})^{(3)}\right]^{\frac{1}{8}}$	$-(j_1+7m-\frac{17}{2})\left[5(j_1+m-\frac{1}{2})^{(3)}(j_1-m+\frac{5}{2})^{(3)}\right]^{\frac{1}{8}}$	$\left[35(j_1+m-\frac{1}{2})^{(3)}(j_1-m+\frac{7}{2})^{(4)}\right]^{\frac{1}{8}}$
$\frac{1}{2}$	$\left[(14j_1+22m+13)(j_1-m+\frac{5}{2})^{(3)}-(j_1+m+\frac{5}{2})(11j_1-13m+\frac{19}{2})\right](j_1+m+\frac{1}{2})^{\frac{1}{8}}$	$\left[(j_1+m+\frac{3}{2})(j_1-3m+\frac{5}{2})+4(m-1)(j_1-m+\frac{5}{2})\right]\left[15(j_1+m+\frac{1}{2})^{(3)}(j_1-m+\frac{3}{2})^{(3)}\right]^{\frac{1}{8}}$	$-(j_1-7m+\frac{19}{2})\left[5(j_1+m+\frac{1}{2})^{(3)}(j_1-m+\frac{5}{2})^{(3)}\right]^{\frac{1}{8}}$	$-\left[35(j_1+m+\frac{1}{2})^{(4)}(j_1-m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$
$\frac{3}{2}$	$-\left[(j_1+m+\frac{5}{2})(j_1-3m+\frac{3}{2})+2(2m-1)\right]\left[(15(j_1+m+\frac{3}{2})^{(3)}(j_1-m+\frac{3}{2})\right]^{\frac{1}{8}}$	$\left[(j_1+m+\frac{7}{2})^{(3)}-10(2m-1)\right]\left[(j_1-m+\frac{5}{2})\right]\left[(j_1+m+\frac{3}{2})^{(3)}\right]^{\frac{1}{8}}$	$(3j_1-7m+\frac{25}{2})\left[3(j_1+m+\frac{3}{2})^{(4)}(j_1-m+\frac{5}{2})\right]^{\frac{1}{8}}$	$\left[21(j_1+m+\frac{3}{2})^{(3)}(j_1-m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$
$\frac{5}{2}$	$-\left(j_1-7m+\frac{7}{2}\right)\left[5(j_1+m+\frac{5}{2})^{(3)}(j_1-m+\frac{5}{2})^{(3)}\right]^{\frac{1}{8}}$	$-\left(3j_1-7m+\frac{21}{2}\right)\left[3(j_1+m+\frac{5}{2})^{(4)}(j_1-m+\frac{5}{2})\right]^{\frac{1}{8}}$	$-\left(5j_1-7m+\frac{35}{2}\right)\left[(j_1+m+\frac{5}{2})^{(3)}\right]^{\frac{1}{8}}$	$-\left[7(j_1+m+\frac{5}{2})^{(4)}(j_1-m+\frac{7}{2})\right]^{\frac{1}{8}}$
$\frac{7}{2}$	$\left[35(j_1+m+\frac{7}{2})^{(4)}(j_1-m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$	$\left[21(j_1+m+\frac{7}{2})^{(3)}(j_1-m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$	$\left[7(j_1+m+\frac{7}{2})^{(4)}(j_1-m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$	$\left[(j_1+m+\frac{7}{2})^{(3)}\right]^{\frac{1}{8}}$

$j_2 = 3$

k \ m_3	0	1	2	3
−3	$-2\left[5(j_1+m)^{(3)}(j_1-m)^{(3)}\right]^{\frac{1}{8}}$	$\left[15(j_1+m-1)^{(3)}(j_1-m+1)^{(4)}\right]^{\frac{1}{8}}$	$-\left[6(j_1+m-2)(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$\left[(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$
−2	$2m\left[30(j_1+m)^{(3)}(j_1-m)^{(3)}\right]^{\frac{1}{8}}$	$-(j_1+3m-2)\left[10(j_1+m-1)(j_1-m+1)^{(3)}\right]^{\frac{1}{8}}$	$2(2j_1+3m-4)\left[(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$-\left[6(j_1+m-2)(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$
−1	$2(j_1^2-5m^2-1)\left[3(j_1+m)(j_1-m)\right]^{\frac{1}{8}}$	$-(j_1^2-10j_1m-15m^2+5j_1+15m-6)\left[(j_1-m+1)^{(3)}\right]^{\frac{1}{8}}$	$-(j_1+3m-3)\left[10(j_1+m-1)(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$\left[15(j_1+m-1)^{(3)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$
0	$-4m(3j_1^2-5m^2+3j_1-1)$	$2\left[j_1(j_1+1)-5m(m-1)-2\right]\left[2(j_1+m)(j_1-m+1)\right]^{\frac{1}{8}}$	$2(m-1)\left[30(j_1+m)^{(3)}(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$-2\left[5(j_1+m)^{(3)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$
1	$-2(j_1^2-5m^2+2j_1)\left[3(j_1+m+1)(j_1-m+1)\right]^{\frac{1}{8}}$	$-(j_1^2+10j_1m-15m^2-3j_1+25m-10)\left[(j_1+m+1)^{(3)}\right]^{\frac{1}{8}}$	$(j_1-3m+4)\left[10(j_1+m+1)^{(3)}(j_1-m+2)\right]^{\frac{1}{8}}$	$\left[15(j_1+m+1)^{(3)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$
2	$2m\left[30(j_1+m+2)^{(3)}(j_1-m+2)^{(3)}\right]^{\frac{1}{8}}$	$-(j_1-3m+3)\left[10(j_1+m+2)^{(3)}(j_1-m+2)\right]^{\frac{1}{8}}$	$-2(2j_1-3m+6)\left[(j_1+m+2)^{(4)}\right]^{\frac{1}{8}}$	$-\left[6(j_1+m+2)^{(3)}(j_1-m+3)\right]^{\frac{1}{8}}$
3	$2\left[5(j_1+m+3)^{(3)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$\left[15(j_1+m+3)^{(3)}(j_1-m+3)^{(3)}\right]^{\frac{1}{8}}$	$\left[6(j_1+m+3)^{(3)}(j_1-m+3)\right]^{\frac{1}{8}}$	$\left[(j_1+m+3)^{(3)}\right]^{\frac{1}{8}}$

$ 1.75

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